Personal Luck
Why premodern China — probably — did not develop probabilistic thinking

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Styles of thought

Why certain mathematical ideas appeared when they did, and why they did not when they didn't, is a well-known puzzle the moment one steps outside the apparent *internal* logic of a topic — which is often not enough in itself to generate progress — and tries to relate intellectual events or non-events to the *external* social world. The more-than-millennial hiatus in the growth of the Babylonian theory of quadratic equations after the early second millennium BCE is an illustration of this.[1]

The present essay is an attempt to open up an equally curious subject. Looking through Li Yan's *History of Calculation in China*,[2] and the works of Needham and Wang,[3] Libbrecht,[4] and of Martzloff[5] on the history of premodern Chinese mathematics, as well as the collection of mathematical-philosophical essays on China edited by Volkov,[6] one is impressed by the virtual absence of anything explicitly relating to the theory of probability. Likewise, Nakayama's *Academic and Scientific Traditions in China, Japan, and the West* makes no mention of probability or statistics.[7] Why does this absence — if absence it is — matter? Alistair Crombie has shown that modern science required the combination of up to six styles of thinking that were, initially, separate. These were the 'postulational', 'experimental', 'hypothetical modelling', 'taxonomy', 'probabilistic', and 'historical derivation'.[8] It can be shown that premodern China possessed all of these, at least in some measure, with the seeming exception of the fifth, the probabilistic mode.[9] It is worth noting that probabilistic thinking defined in terms of a calculus of probabilities was the last of these styles to appear in the West, fleetingly in the sixteenth century with Cardano and then, definitively, in the seventeenth, with Fermat and Pascal. One has the suspicion that it is the style of thought least 'natural' to most human beings. Probabilistic thinking, though studied by some great names in the West such as Huygens, Jacob Bernoulli, Halley, De Moivre, and Laplace, also integrated late with other styles of thought, only moving to the center with statistical mechanics in the later nineteenth century.[10]

Probabilistic thinking, however 'unnatural' it may or may not be, also has certain aspects that should have encouraged an early rather than a late appearance. Its simple forms lead with ease to immediate 'experiment', like tossing dice, that can be both intellectually illuminating and remunerative if the theory followed is a correct one. (Insurance and annuities are other examples.) This makes its late appearance in the West, and its non-appearance or at most its minimal appearance in premodern China, doubly baffling. Martzloff's suggestion that "the insistence on real problems may have constituted a powerful brake" on the development of mathematics in China seems hardly to apply.[11] Since it will be shown here that the *indirect* evidence for some understanding of probability is persuasive, if not quite decisive, it is likely that the pioneers kept their discoveries to themselves in order to monopolize the benefits. Further, though officials were punished from time to time for gambling under many of the earlier dynasties, a specific ban on officials taking part in such activities was first promulgated under the Jin dynasty in 1168, and then continued under the Ming and Qing.[12] This may have inhibited the scholar class in these later times from writing about the topic too easily or openly, and tended to prevent the creation of a domain of public discussion about it.

Routine monetization was well-established in most of China by the eleventh century, and would have made the fine-tuning of payoffs relative to odds easier to conceptualize and to put into effect. Conversely, the Tang code, some centuries earlier, in its laws against gambling, revealingly defined stakes only in the rather substantial units of cloth-rolls (*pi* 疋).[13] Medieval aristocrats sometimes played each other not only for huge sums of money but also for such effectively indivisible stakes as estates.[14] True, Hong Mai, writing on Chinese-style backgammon in 1151 when northern China was occupied by the Jurchen Jin dynasty, could still observe that "the men of the north gamble for gold and silver, male and female slaves, and sheep and horses,"[15] but this applied to non-Chinese invaders. Professional gambling operations — creating what might be called ‘the casino-and-customer dichotomy’ in contrast to the older style of gambling between approximate equals— go back a long way, but are first recorded on any substantial scale in China in the early second millennium, roughly coinciding with the Chinese medieval economic revolution.[16] Thus in the thirteenth century the Yuan code prescribed penalties for institutionalized gaming:[17] “Those who gamble for money or goods shall be given seventy-seven strokes with the heavy bamboo, and the money and goods confiscated.... Those who open gambling houses (*bofang* 博房), shall be dealt with in like fashion. If they repeat the offence they shall be additionally banished for a year.... Those who gamble for drinks or for food shall not be indicted.... Whenever gambling comes to light because of an affray, it shall be traced back to the coin-tossing arena (*tanchang* 攤場).” These sorts of regularized operations ought to have exerted pressure for ‘fair’ outcomes with dice and other devices, which mattered hardly at all in the earlier symmetrical gambling between equals, so long as defects were the same for everyone. Whether they did have this effect is a question still to be investigated: the account given of the late-imperial ‘colored-card meetings’ (*huahui* 花會) in a later section suggests that the point should not be automatically taken for granted.

China also shared much of its gaming technology with the Middle East, India, and Europe. The 6-faced cubic die was almost certainly an import, since it was known much earlier in Western Asia than the fifth century CE[18] when it still had not replaced the Chinese traditional set of five 2-faced wooden throwing-spills (*wumu* 五木), and oddities like the Chinese 18-sided die.[19] Parlett has shown that if the probable outcomes of 10 binary lots are compared with those of two 6-sided dice, the dice produce a higher proportion of extreme values, and so lead to a more exciting game.[20] The Chinese do not seem to have used more than one set of 5 binary black/white throwing-spills, with its limited six possibilities.[21] In later times, however, they normally used more than one die, as is shown later. By the time that Cheng Dachang was writing on the board-game of *chupu* 樗蒲 in the twelfth century he had to give detailed arguments to establish that this change from spills to dice had actually happened, so completely had the collective memory by this later period forgotten the earlier technique.[22]

> In Tang times they incised holes in [cubes of] bone and filled them variously with red and black paste, using numbers [pips] to show the scores, which was surprisingly ingenious.... After incision in bone was used to make dice (*tou* 骰), not only was the old system of the five wooden throwing-spills lost and not passed on, but the written character became 骰 [with a ‘bone’ radical] and no longer 投 [the homophone *tou* = ‘throw’, with a ‘hand’ radical]. Its physical structure was completely different from the time when wood had been used. When wood was employed, the two ends had been narrow and pointed, with the middle section wide and flat, like the shape of a present-day almond kernel. Because the ends were narrow and sharp the spills could roll and jump about. Because of the flat part the scores could be incised on them. Each spill had two faces. One was colored black with a calf painted on it.... The other was colored white with a pheasant painted on it. Usually 5 spills were thrown. If they all came up black this was called an ‘all-black’ (*lu* 盧).... This was the highest score in *chupu*. Once the spills had been rolled between the palms and tossed, a player would often yell at them to make them reach this maximum. This was known as ‘shouting for an all-black’. The next best score was four blacks and one white, in other words four calves and a pheasant, being known as ‘a pheasant’. Lower still was when blacks and whites were jumbled together indifferently, this sometimes being known as an ‘unfilial owl’ (*xiao* 梟). When *The Sayings of Deng Ai* assert that “In the board-game of Sixes (*liubo* 六博) whoever gets an unfilial owl is victorious (*sic*),” he may have meant a ‘gelded bullock’ (*jian* 犍).[23] It used to be said that ‘There is no one who cannot get a ‘gelded bullock’ in 10 throws of the 5 wooden spills’.

The ‘gelded bullock’ was the term for 2 whites and 3 blacks.[24] The probability over 10 throws of *not* getting a 3 + 2 mix, with black having the 3, is $(22/32)^{10} = 0{\cdot}02359$ or somewhat more than 2 per cent. Cheng also quotes a famous if uncritically compiled Northern Song anthology, the *Imperial Survey of the Taiping Xingguo Reign-period*[25] to the effect that “It is possible when playing Sixes to get 5 ‘gelded bullocks’ in 5 consecutive throws.” The probability of doing this is 0·00298, or about 0·3 per cent, some 3 in every thousand tossings of the spills. The point here, though, is not the chance to confirm ancient wisdom with modern calculations, but to show the existence of an interest in frequencies and of a certain intuition that they could have their own inherent laws. Nonetheless the Chinese phrase 'is unable to' (*bu neng* 不能) also carries a slight implication that the player must in some sense be 'trying' for this result. Cheng goes on to say: "The 5 throwing-spills had only 2 faces, whereas dice have 6. Since they have 6 faces, they can also show 6 numbers, which are not comparable to the mere 2 faces, black and white, of the 5 throwing-spills."

In Yan Zhitui's *Family Teaching of Mr Yan*, written in the sixth century CE, there is a comment that likewise suggests a taste for complexity that grew over time. Referring to two ancient board-games that used two different types of aleatoric device, probably spills and a pair of some sort of dice, perhaps 18-sided,[26] he declared that "the techniques of calculating used in them are so shallow and limited that they do not afford one entertainment."[27]

An anthology of miscellanea that appeared in 1735, *The Source-Mirror of Empirical Investigation*, quotes a passage on dice from an earlier anthology to the effect that "In [ancient] the board-game of Sixes they tossed six spills to move six pieces [per side]. *At this time dice had not yet made their appearance.* These must date from the Wei and Jin dynasties [third century CE] at about the same time as the board-game of Grip Lance (*woshuo* 握槊)."[28] Another source cited in *The Source-Mirror*, presumably from the middle of the first millennium, describes Grip Lance as "a barbarian game that has recently entered China" and notes that it was flourishing in the third century CE.[29] This is scrappy evidence, but suggestive of a similar foreign origin for cubic dice.

That they were a novelty to many people in Tang times is clear from an essay by Liu Yuxi composed in the middle of that dynasty, "Watching Gambling":[30]

> Among the guests was one who devoted himself to dice-and-board games, and invited me to watch. Previous to this, our host had placed the equipment for Grip Lance under the covered verandah and remarked, "The host makes the introductions, those invited form bonds with each other, and he then bows in self-deprecation and retires to a secondary position." The devices for allocating points in the games (*bochi* 博齒) were different from those of past times. They were made of bone cut into a four-square shape and all inlaid with red and black [dots]. The numbers [on opposite faces] added in pairs to the same total.... One observed how they rolled and then came to a standstill, after which one relied on the result to struggle on one's way [round the board].
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> On this day, the guest tossed the bones onto the board and prayed to them, saying "May you come as in haste, and depart as if making your escape. May you be unable to move before the moment has arrived, and be without error in fulfilling my commands! Nor shall you follow the calls of others, nor disdain my distress!" Those present divided themselves into pairs and sat pressed tightly together from the rising of the sun until it was declining in mid-afternoon, but the final outcome was not what he had prayed for. *This guest was persuaded that the bones had feelings in them,* and it may have been that he had been relying on this. So he did nothing but curse them for not yielding their secrets. Then he also bit them and trampled on them, paying no heed to the mockery and reproaches of the onlookers. This done, he asserted, "It was not that my techniques were not well-crafted, but that this decayed skeleton would not give me anything."

The self-centered superstition of the gambler and the cool and observant skepticism of the observer make a characteristically Chinese contrast.

The popular dice-and-board game Double Six (*shuanglu* 雙陸), in its various Chinese forms, was a version of backgammon, or a prototype of backgammon, that was imported from the regions west of China. Hong Mai wrote that "it began in western India,[31] and flowed in during the Wei dynasty of the Cao family,[32] flourishing during the Liang, Chen, Qi, Sui, and Tang dynasties."[33] The somewhat shadowy *chupu* (樗蒲), kin to Pachisi/Parcheesi and Ludo, sounds as if it might have been the Indian dice-and-board racetrack game Chaupar.[34] In return, twelfth-century China seems to have invented the earliest dominoes.[35]

Games of chance thus traveled easily in Eurasia. This suggests that any gap in mentalities in *this* domain as between the eastern, middle, and western parts of the civilized Old World are unlikely to have been great. We have of course to bear in mind Feyerabend's dictum that "the discovery and development of a particular form of knowledge is a highly specific and unrepeatable process,"[36] while not being mesmerized by it. Some of the processes that develop knowledge are robust, in that quite a wide variety of inputs does not affect the outcome. As will be shown later, the taxonomic classification of multi-dice combinations was probably one of these. Others are critical processes in that, so far as we can tell, quite small variations in the inputs would probably have caused them to abort. The origination of probabilistic thinking may have been critical in the sense just defined. If this is so, then the task will be to identify the vital variable elements. The relative closeness of many Chinese, European, Indian, and Islamic gambling practices, and the historical sharing of a substantial part of the technology, suggests that broad-brush explanations are likely to explain little.

It is also worth stressing that technical 'success' or 'failure' in basic probability theory allows for little variation between cultures. The formulae and the figures required are to an unusual degree either just 'right' or 'wrong', inexorably determining winners and losers. Until the contentious problems related to conditional probabilities appeared, arising historically from Bayes's theorem which first became publicly available in the West in the middle of the eighteenth century, the verdict on validity or otherwise is as straighforward as such a judgement can ever be.

Assuming that the problem in the Chinese case does not arise from overlooking crucial texts, which remains a possibility given the huge corpus of Chinese material, the crux from a point of view based on the analysis of 'external' factors is that China was well-endowed. There were long-lived systems of divination that had strong associations with underlying elements of probabilistic thinking. There was a passion for gambling in a variety of forms, some of which both require probabilistic analysis to identify optimal strategies, and are simple enough for this to be possible without advanced theory or computers. There were extensive demographic and other statistics collected by the national and local governments, some of the local demographic ones so good that it has proved possible to construct smooth age-specific mortality curves from them without any averaging.[37] In late-imperial times there were insurance businesses for ventures like trading up and down the coastal seas, requiring at least a minimum of actuarial acumen. There was also some interest in proto-epidemiology.[38]

Probabilistic thinking requires the availability of repeating patterns that can be expressed numerically. Premodern China had an abundance of these in the form of stochastically determined oracles, dice-casting and analogous phenomena from gaming, various records of ages at death, observations of the time and space covariants of diseases, and a commercial concern with the frequencies of disasters. That this abundance was not in itself a *sufficient* cause for the emergence of a calculus of probabilities sharpens the questions that must be asked about China, and also suggests the inadequacies of exclusively external-factor analyses of the rise of probabilistic thinking in Europe.

The concept of 'frequency' was stablished in its simplest form by Tang times, and very likely much earlier. *Providing For Leisure Time* by the ninth-century author Li Kuangyi contains a passage that is of importance in this regard.[39] A translation is therefore offered here but with the warning that parts of it are speculative:

> As regards fortune-telling, old women are not diviners with the tortoise-shell or divining-stalks. They invariably offer opinions on the basis of [leaves from] mulberrry trees or shrubbery. There was an old lady who knew nothing about anything, but set up a stall open to the public where she followed the mulberry [leaves], but predicted *the opposite* of what they showed. One might describe this as [being as ridiculous] as 'selling oracles to a Buddhist monk'.[40] Her inspiration was to *reverse* the sense, so that if the mulberry spoke of good fortune, she would speak of ill, and if the mulberry predicted ill luck she would predict good luck. But in the end the mulberry and the old woman had each of them predicted half of the cases correctly and half wrongly. Some people have judged this to be an instance of 'Guan Zhong being enlightened by the old lady at his gate' [who was reliably wrong], but this is a complete error.
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> It should be noted that *The Spellmaster* says that there was a man from the state of Qi who was good at taking tortoise-shell oracles, scoring a success rate of 5 out of 10. One of his neighbors, who was not good at oracle-taking, routinely predicted *the contrary*, but also was successful 5 times out of 10. This was the same as if no oracle had been taken. — It should be explained that the foregoing was the hypothetical exposition of a principle by a master, but that his successors talked about it so clamorously that in their folly they wished to ascribe reality to it. So they attested that this had actually happened.

We have here not only the idea of frequency but also that of the *complement* of a probability within an exhaustive domain of possibilities. There is also the hint of an attempt to assert the *logic* of probability against supernatural explanations in the refusal to accept that Guan Zhong's case could be relevant.

Another example illustrates a practice that could have given rise to a theory but did not. In Northern Song times, we learn from a record reproduced in a late Ming source,[41]

> Vendors of sugar candies in the capital make a basin that may be more than 3 feet [in diameter]. On it they paint the forms of several hundreds of birds, fishes, and other objects. The length of these is not more than half an inch, with a width of one's little finger. The smallest of them is barely bigger than a couple of beans. That the birds have feet, the shoes laces, and the bows bowstrings, indicates the cunning of their meticulous delicacy. Most of them are like this.
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> They make needles into darts, distinguishing them with feathers of different colors. When the vendors make the basin rotate, the purchasers deposit a coin, and then shoot a dart at it. Those who hit something obtain a sugar candy. As numerous darts rain down on it together, the basin spins without stopping. The candy vendors sing out such cries as 'A hit for white!', 'A hit for red!', and 'The others have all missed!'
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> Once the rotation has ceased and the basin has come to a standstill, they check to make sure there has been no mistake. The vendors then take up the darts themselves, set the basin spinning again, and shoot at it so as to match what the winners have hit. If one of these previously hit the foot of a bird, they will not permit themselves to hit the wing, and if another previously hit the cord of a bow, they will not permit themselves to hit the bow-tip. There is not the minutest deviation from what the winners previously hit.
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> The legendary archer Ji Chang contemplated a louse for three years until it seemed to him to be as large as a cart-wheel. The explanation of the vendors' skill must be of the same nature. But suspending a louse gives one something stationary to look at. In the present case they can look at something rotating and hit it. Doesn't one have to acknowledge that something at rest is easy, but something in motion hard? It may be that they follow a particular Way [that enables them to do what they do].

We can assume, for the purpose of estimation, that each painted item filled a square with sides of 0.5 inch, and that there were 400 such items. The Chinese inch is 0·1 of a Chinese foot, and the total surface area of a shallow circular basin 3 feet in diameter is close to 7 square feet. This area thus contains $20 \times 20 \times 7 = 2800$ squares $0{\cdot}5 \times 0{\cdot}5$ inches in size. If the purchasers threw their darts *at random*, but without missing the basin altogether, they would have had 1 chance in 7 of hitting a target item and winning a candy. The spinning basin, for unskilled dart-throwers, was an aleatoric device, though this aspect of its nature may have been hidden by the vendors' skill.

This example, seen through our eyes, models a *probability space*. The *proportion* of the total area respresented by the target area defines the probability of winning, which is about 0·1429. The vendors would have known what proportion of purchasers won and what proportion lost over the long run, having a very large sample to allow them to determine this, and their accounts as a form of record. They must have had some intuition that this proportion was approximately the same as the *ratio* of the area taken up by target designs to the total area of the basin surface, and that the proportion of winners could be altered *pro rata* by altering the ratio occupied by the target area. They were thus actually looking at a direct quantitative representation of the probability without, at least consciously, seeing it in such terms. Their failure in this respect is in turn a representation of *our* problem.

It is also relevant to ask to what extent premodern Chinese mathematics had a formal concept of an *statistical distribution*.[42] From at least early-imperial times the Chinese could calculate a simple arithmetic mean. It appeared as the by-product of the operation called 'equalizing allocations' (*pingfen* 平分), and it is perhaps appropriate to call this as an 'arithmetical' as opposed to a 'statistical' average in that it was not thought of as a measure of central tendency. The reason for this was that 'equalizing allocations' removed the pattern of a distribution rather than being a part of the process of characterizing it. Thus problem 16 in the first chapter of the *Nine Chapters* reads:[43]

> Again, we take as given 1 part out of a division into 2, 2 parts out of a division into 3, and 3 parts

> out of a division into 4. We ask: if we reduce the excessive and augment the insufficient, what is the quantity for each of these that will bring it into equality (*ping* 平) with the others [similarly treated]?[44]

This requires determining the mean of 1/2, 2/3, and 1/4 as an intermediate step.The answer supplied is: “Reduce the 2 parts of a division into 3 by 1, and the 3 parts of a division into 4 by 4, and add these to the 1 part of a division into 2, after which all of them will be equal at 23 parts of a division into 36.” We would understand this as

$$\frac{18+5}{36} = \frac{24-1}{36} = \frac{27-4}{36} = \frac{23}{36}$$

which is the mean required. The method by which the *Nine Chapters* reaches its result may be summarized as follows: (1) The three fractions were added together in what is today the usual manner. That is, each dividend was multiplied by the denominators of the other fractions, these results summed, and then the sum divided by the product of all the denominators. This gives 46/24. The mean would have been derivable from this by dividing it by 3, the total number of items, to yield 46/72. This not stated, perhaps because it was thought to be obvious. The objective in any case was different, namely *redistribution*. (2) Each of the three products of a dividend with the other denominators (that is, 12, 16, and 18) was multiplied by the number of items, here 3. The same was done for the product of all the denominators (24). The *differences* of each fraction from the mean were then determined, giving

$$\frac{36-46}{72} = \frac{-10}{72}, \quad \frac{48-46}{72} = \frac{+2}{72}, \text{ and } \frac{54-46}{72} = \frac{+8}{72}.$$

We would think of this today as the common operation of calculating $x_i - \bar{x}$. (3) The signs of the differences were reversed in order to “reduce the excessive and augment the insufficient,” and the expressions simplified to the lowest common denominator. In the third chapter *weights* are also used in redistributions of this sort to create the appropriate sort of equity (not equality) to allow for differences in such characteristics as socio-political status or, for animals, species.

The problems that derived from the need of the imperial bureaucrats to make the burden of taxes and labor-services ‘equitable’ (*jun* 均) also seem at first sight to provide cases of statistical distributions. An example can again be found in the *Nine Chapters on the Mathematical Art*.[45] Four ‘counties’ have to share equally the burden of moving 250,000 ‘bushels’ of grain to a given place using 10,000 carts.[46] The populations in households and the required travel-times in days are shown in Table 1:

TABLE 1

The equitable allocation of ten thousand transport carts

County	Households	Days Travel	*h*/*d*	(*h*/*d*)/∑(*h*/*d*)	Carts (rounded)
A	10,000	8	1250	0·3324 47	3324
B	9,500	10	950	0·2526 60	2527
C	12,350	13	950	0·2526 60	2527
D	12,200	20	610	0·1622 34	1622
Totals			3760	1·0000 01	10,000

In what numbers should carts be assigned to each county?

Formulating it in our terms, we assume that each cart carries 25 bushels and makes exactly one complete single trip. The fourth column, *h*/*d*, shows the number of households that would be available to provide the support for one cart.day of *one* complete multi-day trip of the time required for each county. The fifth column shows the proportion of the total of all the entries in the fourth column that is available from each county. Since the more households available per cart-day the lighter the burden per household, this is counterbalanced by assigning *pro rata* more of notional single cart to the counties with more disposable resources, as shown in the fifth column. Since 10,000 carts are available, the final column is derived by multiplying the entries in the preceding column by this number, and then rounding, as carts are integral objects. The burden of cart.days per household, *cd*/*h*, is now close to identical at about 2.6 for all the four counties. (Thus for A, one uses ((3324) × 8/10,000), etc.)

The answer given in the *Nine Chapters* is the same as in Table 1, and the method used is similar. That is, it calculates the *h/d* values and uses the ratio of each such value to the sum of the values as a set of weights, *which sum to unity,* though it does not calculate the *cd*/*h* ratio. What is striking is that the operation is again aimed at *neutralizing* the effects of the unequal distributions of populations and travel-times. The concept of 'equity' is also not the same as that of 'average'. Nonetheless, this dataset would appear to us to lend itself to further questions. What, for instance, is the burden in cart.days per household in each county if the carts are assigned in equal numbers per county? This works out as 2, 2·63, 2·63, and 4·1 respectively. Again, how does one measure the degree of *variation* that this last distribution expresses? We might say that $\sigma_{pop} = 0{\cdot}77$. Or that the mean burden per household under equal cart allocation is 2·84 unweighted, and 2·89 weighted by proportion of population. And so on. Some bureaucrats must have thought at times about such varying measures, even if not necessarily in these terms. What has yet to be found is any evidence that they wrote about them. Overall, it is tantalizing to see how close Chinese mathematicians more than a millennium-and-a-half ago came to dealing with statistical questions without, it seems, thinking in the least statistically.

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The survey that follows is designed only to open a discussion. If there are any tentative conclusions they are the following. (1) The conceptual crux was the inability to invent the appropriate mental models. (2) The technical crux, judging from publicly available material, was a failure to *link combinations and permutations*, both of which were separately familiar by mid- or late-imperial times. And (3) some simple probabilistic thinking did in fact exist in premodern China, but like an underground source of water. In other words, it was normally invisible, but betrayed its existence at certain moments in the way that a spring can indicate a subterranean source.

Analyzing the history of mathematical ideas is in one sense a perverse undertaking. Whereas the mathematician strives to make what is obscure clear, and what is difficult easy, the historian has to attempt the reverse process. He or she must re-create in his or her imagination the state of mind in which what is now well understood was once bewildering and opaque. Otherwise the difficulties will appear too trivial to be comprehensible. Sample calculations have been included in what follows so the reader can form a more specific impression of where opportunities existed and obstacles lurked. But a caveat is necessary: the use of modern notation and terms makes the challenges seem easier than they would once have been.

Conceptual models

The conceptual breakthrough needed for probabilistic thinking is a clear idea of the number of *distinct* ways in which a given outcome can occur. In the simplest non-trivial examples, a single coin can land either heads or tails when tossed, or a randomly chosen integral number be either odd or even, in both cases with equal likelihood. There is only *one path* to each result. This principle was sketchily understood in medieval China. In the *Family Teaching of Mr Yan*, Yan Zhitui, who lived under the Northern Qi dynasty in the sixth century CE, and had a certain skepticism toward divination in his own degenerate day, insisted that the proper *test of the significance* of a result depends on comparing it with what would have been *expected* under 'normal' circumstances, using this simple case as an example.[47]

> Divining by means of the cracks in a burnt turtle undershell and by the manipulation of the stalks of plants was a business undertaken by the sages. In recent times, however, it has failed to regain the excellence that it once had. Most of the masters of the art are incapable of hitting the target. In

> ancient times, people practised divination to resolve their doubts. These days people's doubts arise *from* divination. If someone makes plans properly and in good faith, but then obtains an unfavorable hexagram[48] through divination, this will on the contrary cause him to be panic-stricken. What is the sense of this? If someone gets divination right *six or seven times out of ten*, he is accounted an expert, but there is nothing exceptional about having a rough idea of how things are moving. The general rule is that if one is guessing whether an odd or an even number [is hidden under an upturned bowl], *it is natural that one gets a score of one half.* How can one take [sixty to seventy per cent success] as enough to depend upon [among diviners]?

Popular instinct was closer to the mark than Yan allowed. If the diviners guess for 25 trials each, about 21 per cent will have scores 60 per cent or better on the assumption that the outcome is pure chance. This drops to about 5 per cent for scores of 65 per cent or better, and to about 2 percent for scores of 70 per cent or better. If the number of trials per diviner is raised to 50 each, these percentages drop to about 10 per cent, 1.6 per cent, and 0.6 per cent respectively. For 100 trials each, we have about 2.6 per cent, 0.1 per cent, and virtually 0.[49] We can thus assume that Yan did not make his remarks on the basis of systematic observation or analysis, but only of casual impressions.

There is only one pathway for a once-off guess 'odd or even', as there is for the toss of a single coin. As the previus example shows, the question stops being trivial when the guessing or tossing is repeated, as there is more than one pathway to an outcome. Tossing 2 coins yields 4 pathways to 3 distinct combinatoric outcomes: HH, TT, and (TH and HT). The probability of a head-and-tail combination is therefore double that of one of two heads or of two tails. There is likewise a probability of 1/6 for each score when tossing a fair die with 6 faces marked '1' to '6'. However, when 2 dice are thrown together, or in sequence, are combined scores of '11' and '12' equally likely? Most people today will correctly say that '11' is twice as likely as '12' as there are 2 pathways to '11', namely '5, 6' and '6, 5', while there is only one to '12', namely '6, 6'. This may also seem trivial, but it is not. Three hundred years ago, the great mathematician Leibniz got the answer to this question wrong, thinking the odds equal.[50] His failure suggests that the difficulty of the problem is now hidden from us because of our modern cultural conditioning.

How near, then, did premodern Chinese come to path-analysis? As so often with China, the short and frustrating answer is 'almost'. Consider a passage from the *Notes from the Garden of the Brooklet of Dreams* by the 11th-century polymath Shen Gua.[51] It concerns 'encircling chess' or *go*. This is game in which 2 players alternately place black and white counters respectively, drawn from a theoretically unlimited supply, on any of the currently vacant intersections of the 361 on a board ruled with a grid of 19 × 19 lines. These pieces are not moved, but can be 'captured', that is, removed from the board, when a contiguous group of one or more is surrounded without any gaps by enemy pieces (and the edges of the board where this latter is relevant).[52] The winner is the player at the end with the largest number of such captives plus empty intersections surrounded by his own forces.

> According to those who write on strange and miscellaneous matters, Yixing, the monk of Tang times, calculated *all the numerical possibilities* for the placement of pieces on the board of encircling chess. That is, he exhausted the number of arrays on the board. When I thought about this, I realized it was assuredly only a simple exercise, but the magnitude of the number is too great to be expressed by the numerals used in our world. I will now give a summary account of how one reaches this large number:
>
> If one considers a 2 × 2 linear grid [with 4 intersections] and uses 4 pieces [of each of the 2 colours], there are 81 possible transformations of the board.

Note that since each intersection can be (1) empty or (2) covered by a black counter or (3) by a white one, the total *permutations* are $3^4 = 81$. Shen Gua continues by induction:

> With a 3 × 3 linear grid and 9 pieces, there are 19,683 possible transformations of the board [3^9]. With a 4 × 4 linear grid and 16 pieces, there are 43,046,721 possible transformations of the board [3^{16}].... For a 7 × 7 linear grid and above, we have no numerals that can record such large numbers. For the grand total of a 361 × 361 linear grid [he means 19 × 19] one approximates it by writing a sequence of 10^4s multiplied by themselves 52 times. This is the grand total of possible arrangements

of the board.

His method is right but his answer is too large. The correct value of 3^{361} is close to $1{\cdot}74 \times 10^{172}$, which is of the order of 10^4 multiplied by itself 43 times, not 52. He later justifies his method by saying:

> The first intersection can manifest 3 transformations [empty, black, or white]. Thereafter one ignores the grid structure and merely adds another counter which implies three-branched consequences, making 361 such additions in all, *each of which has three-branched consequences*.

Shen had no fear of huge numbers. And he saw that a complex problem is often best tackled by means of a method developed from analogous simpler versions.

Prolog to analysis

Is our quest intrinsically mistaken? In the well-known words of Mencius, "like climbing a tree to catch a fish"?[53] We need to face the possibility that we are inappropriately looking for modern Western ways of thinking in a non-modern, non-Western culture, and ought not be be surprised that they are not there. I think that this is a false sophistication in the present case. Here, in outline, is why:

Consider the plebeian ancient game of Lay Out The Coins (*tanqian* 攤錢).[54] It had a number of other names as well: Mind Money (*yiqian* 意錢), Guess Numbers (*sheshu* 射數), and Tricky Estimate (*kuiyi* 詭億) among them.[55] In the later nineteenth century it was the basis for *fantan* 番攤, played in state-supported dens to provide finance for the navy.[56] By this late date at least, the game was played as follows:

An arbitrary quantity of copper coins or counters, was drawn at random from a heap called 'the cash surface' (*qianpi* 錢皮), and put into a receptacle (the *tanchong* 攤盅). The participants then laid bets on the residue that would remain when the contents of the receptacle were later counted off by fours. (Thus 30, which is (4 × 7) + 2, would leave a remainder of 2.) The 4 possibilities, or 'gates' (*men* 門): 1, 2, 3, and 4 or 0, were taken as a cycle for purposes of ordering. There seem to have been four types of possible wager:[57]

1. *Fan* 番. Betting on a *single* number. The payoff for success was the stake back plus 3 times the stake.
2. *Nian* 捻, or *ren* 稔. Betting on 2 *differentiated adjacent* numbers, one of which was the *principal* and the other the *support*. If the winning number was the principal, the payoff was the stake back plus 2 times the stake. If the support was the number that came up, the stake was returned.
3. *Jiao* 角. Betting on 2 *adjacent* numbers *treated equally*. If either was the number that came up, the payoff was the stake back plus 1 times the stake.
4. *Zheng* 正. Betting on 3 *sequential* numbers, the one in the *middle* being the principal. The payoff for the principal was the stake back plus 1 times the stake. If either of the other 2 numbers came up, the stake was returned.

The gambling establishment derived its income from a 10% levy on winnings. This apart, is it a fair game? It seems so. Writing '*s*' for 'stake', we have the following expectations for the 4 options:

1. $(1/4) \times 4s = 1s$
2. $(1/4 \times 3s) + (1/4 \times 1s) = 1s$
3. $(1/4 \times 2s) + (1/4 \times 2s) = 1s$
4. $(1/4 \times 2s) + (1/4 \times 1s) + (1/4 \times 1s) = 1s$ (1).

In the case of the first option, *fan*, this impression of fairness is obviously correct. In the case of the other 3 options, it is necessary to consider the effect of bets excluded by the rules governing the relative positions of the numbers chosen. Putting the principal number in bold type, where relevant, we have

2. Permitted: **1**2, 1**2**; **2**3, 2**3**; **3**4, 3**4**; **4**1 and 4**1**. *Not permitted:* **1**3,1**3**, **2**4, and 2**4**.[58]
3. Permitted: 12, 23, 34, 41. *Not permitted:* 13, 24.
4. Permitted: 1**2**3; 2**3**4; 3**4**1, 4**1**2. *Not permitted:* **1**23, 12**3**; **2**34, 23**4**; **3**41, 34**1**; **4**12, 41**2**.

In other words, in options 2 and 3, only two-thirds of the possible bets are allowed, and for option 4, only one-third. Computer simulation and analysis both show the debarred options do not affect fairness. The number of coins drawn can have a slight effect on the frequency of the 4 moduli, but simulation also shows this can be virtually eliminated by requiring the number of cash drawn to fall within a reasonable range. In 10^7 trials for the range 31 to 98, all 4 moduli appeared at a frequency of 2.50×10^6. Analytically, the irrelevance of the restrictions to the outcomes is suggested by the observation that, given the payoffs tabulated above, a gambler limited to betting on a unique number or unique pair or triple of numbers still breaks even so long as they occur with a fair frequency. A detailed analysis is given in Appendix A.

It may have been enough for the construction of this game to see that, given numbers occurring with equal probability, the better's net gains and losses sum to zero for each of the four betting patterns. These may be expressed as the four cyclical groups respectively formed from (-1, -1, -1, +3), (-1, -1, 0, +2) *and* (-1, -1, +2, 0), (-1, -1, +1, +1), and (-1, 0, +1, 0). The structure of the game nonetheless embodies in *implicit* form four of the principles needed for an elementary calculus of probabilities: (1) The *equal probability* of a number of outcomes. (2) The *addition* of *alternative* probabilities. (3) The *exhaustive* consideration of all the possibilities. And (4) the determination of the *expectation* as the *product of the probability and the payoff*. The multiplication of sequential distinct probabilities is, however, missing. Since Lay Out The Coins seems to have been an old indigenous Chinese game, direct foreign influence appears unlikely. But we only have the crucial detailed information of its workings for the very end of the imperial period, and thus we have to bear in mind the possibility that the structure was refined around this time, conceivably as the result of outside influence.

The multiplication rule was however implicit in various traditional coin-tossing games, typically employing 3, 4, 6, or 8 coins. In a game like *guanpu* 關撲 these coins were treated analogously to dice. Chinese coins had a 'heads' side, most commonly called *zi* 字, and a 'tails' side, normally known as *mu* 幕. Bets were made in two main ways: (1) on exactly how many heads or tails would be showing, and (2) on which pattern would appear. For example, for this second type of game, one possibility was that all the coins were either all heads or all tails, or else that, with an even number of coins, there were exactly half of each. As will be shown in a later section, these patterns were analyzed and classified by type in late-imperial China.

As is common knowledge, for 3 coins the outcomes, ignoring different orderings, would tend to the limits:

HHH	$(1/2)^3$	$= 1/8$
THH	$(1/2)^3 \times 3!/2!$	$= 3/8$
TTH	$(1/2)^3 \times 3!/2!$	$= 3/8$
TTT	$(1/2)^3$	$= 1/8$

where the '!' indicates the factorial $1 \times 2 \times 3 \times ... \ n$. The numerators of the fractions (1, 3, 3,1) can also be found from Pascal's triangle.[59] Chinese mathematicians had been familiar with this array since at least Song times, but do not seem to have used it for the analysis of probability.[60] For patterns, using the two examples given above, and assuming 4 coins, HHHH or TTTT would have a frequency tending toward $(1/2)^4 + (1/2)^4 = 2/16$, and HHTT, which has 6 possible orderings, would have a frequency tending toward

$$(1/2)^4 \times \frac{4!}{2! \times 2!} = 6/16 \qquad (2).$$

As of now we have no historical records telling us the payoffs. It is reasonable, though, to suspect that the empirical frequencies were known by some people to a good approximation. There are references to Song-dynasty shopkeepers using some version of *guanpu* to boost demand for their wares. If the customer won a game with the shop-owner, he got back his stake and also the goods free of charge. If he lost, the shopkeeper kept the stake, and the customer departed with no goods. The term for this practice was 'sale by gambling' (*pumai* 撲賣, or *bomai* 博賣). Only shopkeepers who had a good notion of the odds would have stayed long in business.[61] But there is no reason to believe that they *must* have known *how* these odds could be calculated theoretically.

The oldest reference to aspects of the technique of Lay Out The Coins is in the little anthology *Providing For Leisure Time* by the ninth-century author Li Kuangyi.[62] The item is entitled "Coin Games" and the key passage states that "some coin games always take 4 coins as a set," and identifies them with Mind Money mentioned above as a name for Lay Out The Coins. The coins must not be put into matching stacks (*dieying* 疊映), which would amount to cheating. "These days," he grumbles, "everyone writes of this game with coins as *chupu*.[63] What greater insult to *chupu* could there be!" The names should not be mixed up in this way.

A still earlier reference, but without any technical information, occurs in the biography of Crown Prince Zhaoming, the elder son of Emperor Wu of the Liang dynasty in the first half of the sixth century CE:[64]

> The Crown Prince had a nature marked by fellow-feeling and a wish to treat others as he would have wanted to be treated himself.... He...once saw some servitors[65] attached to the private apartments at the rear [of the palace] playing at Coin Spreading (*tanxi* 攤戲). This was later the subject of a criminal accusation of gambling, the penalty for which was lifelong banishment with forced labor for officials and document-certified temporary banishment with forced labor for commoners. The Crown Prince declared, "To amuse oneself with one's own money does no harm to public goods. These categories are too severe." He commanded that the penalty be applied for only three years, and that officials be stripped of office.

We do not know exactly how the servitors were playing, except that it was not in a casino context. It would thus be useful to know how they handled the function of the 'bank', which is needed to absorb the short-term ups and downs of fortune. In much later times there seems usually to have been a 'coin-gaming boss' (*tanzhu* 攤主), who presumably dealt with these matters.

The poet Du Fu, writing around 766 on the Kueizhou region in the upper Yangzi, indicates that the game was played at this time by merchants:[66]

> Hemp from Sichuan, and salt from Wu, have longtime been exchanged
> In river-craft carrying bulky cargoes, but swift as the winds on their way,
> To the sound of shanties skilfully sung by agèd pole-wielding sailors,
> While the merchants pass their daylight hours in coin-guessing through the deep waves.

It was a venerable form of amusement.

Song society was besotted with gambling. Meng Yuanlao's *Dream of the Glories of the Eastern Capital*, which describes Kaifeng shortly after the Northern Song government had had to flee south, has this to say of the practice of gambling-purchase mentioned above:[67]

> On the first day of the first month of the lunar calendar, the prefectural government of Kaifeng permitted three days of purchase by gambling (*guanpu* 關撲).... In the city quarters and blocks of dwellings people would chant out that such goods as foodstuffs, tools, fruits, firewood, and coal were available for *purchase by gambling*. Multi-colored covered stalls were put up side by side in places such as Horse Walk and Pan Mansion street, and the raised paved causeways outside the Song Gate on the eastern side of the capital, and Liang Gate to the west, and also outside the Fengqiu Gate to the north, and in the district south of the capital. Here were spread out on display headgear, combs, pearls, jades, ornaments for the head, clothing, artificial flowers, curios and playthings. Dance-floors were laid out between them and also halls for singers, while carriages and horses chased after each other in criss-crossing fashion. Towards evening the women from families of high social standing enjoyed gambling-purchasing (*guanbo* 關博) in an uninhibited manner, joined the audiences at entertainments, and drank and feasted in the restaurants in the markets. This had become so customary that no one tittered with surprise at seeing it. Three days of gambling-

> purchasing were likewise permitted at the Cold Food Festival in the spring and at the winter solstice.

Acording to the eighteenth-century writer Li Dou, there were no restrictions on time and place for the practice under the Southern Song.[68] Hong Mai, writing under this dynasty, thus tells us of a certain Li Jiangshi, that[69]

> It so happened that vendors carrying Yongjia yellow oranges went past the door, and he shouted out in a lively voice that he would gamble for them. After losing ten thousand cash, the irritation showed on his face. "I have lost ten thousand cash," he said, "without a single orange passing my lips."

'Ten thousand' here presumably means only 'far too much'.

A hint of the methods used can be gleaned, but only precariously, from a tale recounted by Zhang Zhongwen that relates to the year 1217.[70] It tells how Zheng Fuli, a medical specialist who was one of the followers of Kong Wei, the magistrate of Gaoan in present-day Jiangxi province, abducted his patron's daughter. The note in parentheses is in the original.

> Several days before this happened, Zheng saw people in the provisional capital at Hangzhou gambling-buying chickens. (The rate at the provisional capital was that 3 copper cash were thrown, 10 *chun* 純 obtaining the chicken and the 3-cash stake back.) This prompted him to gamble to decide whether or not he would do it. He prayed to the dicing coins (*touqian* 骰錢) that he would obtain a *chun cheng* 純成, wishing to abduct this girl. He forthwith obtained a *chun zi* 純字. Putting his reliance on prayer yet again, he gambled a second time, getting another *chun*. He thereupon followed his reckless lusts, and, taking advantage of Mr Kong having to offer a sacrifice at the Yazhai Altar, abducted her that night.

The most plausible *mathematical* interpretation of the technical terms in Chinese is that the 3 coins that constituted the stake for each attempt were thrown 10 times, and that to win they had to show a 'pure pair' each time, that is, no triples. The Kangxi dictionary and Couvreur's dictionary give a rare meaning of *chun*, pronounced *qun*, as 'a pair' of calculating rods, so this is a tenable lexical reading. The probability of no triples in 10 throws of 3 coins is $0{\cdot}75^{10} = 0{\cdot}0563135$. If the price in cash of a chicken is c, then, from the gambling-purchaser's point of view fair values in the equation for the expected return r would give

$$\text{EXP}\,(r) = 0{\cdot}0563135 \times (c + 3) = 3 \qquad (3).$$

and so c, the price implied, is about 50·27 cash. From the point of the seller, this is the upper bound of the acceptable cost of his acquiring and selling a chicken, anything higher involving a loss to him from his payouts over the long run. In practice he would have added a substantial profit margin into this price, which would also have avoided the need to calculate the odds with any precision. Over the long run, about 1/0·0563135 = 17·76 attempts at 10-*chun* throws would have been needed on average for a gambler-purchaser to win one chicken and recoup his stake by mean of a successful attempt.

The seller, for his part, must have had some way of at least *estimating* the probabilities. Provided that either calculation or experience showed him that, on average, for each throw of the 3 coins, one quarter of those doing the throwing lost and were eliminated (HHH and TTT being together one quarter of the 8 possible outcomes), it would not have been hard to see that if, say, 64 people started throwing together, after one throw of 3 coins each, 48 of them would be left, after 2 throws, 36, and after 3 throws only 27, with a quarter being eliminated each time on average. (He would not have needed to know that $27/64 = 0{\cdot}75^3 = 0{\cdot}421875$, etc.) This method of deducting a quarter 10 times would have indicated that, approximately, 3·6 out of the 64 starters would finally win a chicken and their stake back. The seller's initial takings in stake-money would have been $64 \times 3 = 192$ cash. The sum of 3·6 chickens at 50·27 cash each and 3·6 returned stakes of 3 cash each is approximately 181 + 10·8, effectively 192, as it should be. This sort of approximation must almost certainly have been done. Given the variety of goods sold by the gambling method, it also seems a reasonable guess that rules of thumb existed for adapting the figures to different underlying cost-prices. If so, the key problem shifts: why did no one analyze these numerical techniques, generalize them, and eventually publish them?

The alternative interpretation of *chun* as 'pure' *triples* of either 'heads' or 'tails', allowing either of the latter to count on any given throw, gives a probability of success of $0{\cdot}25^{10} = 0{\cdot}00000095$, close to 1 in a million, which is out of the question as a commercial arrangement. It would also imply that a chicken cost over 3 million cash. Unfortunately the few sources that we have seem to suggest that both *chun* and *cheng*, as for example in *hunchun* 渾純 'full pure', and *huncheng* 渾成 'full complete', were in fact associated with complete sequences of either heads or tails.[71] *Chun zi* 純字in the passage just translated certainly looks like 'pure heads'. The beginning of the discussion of gambling-buying in Li Dou's *Painted Barges of Yangzhou*, written in the eighteenth century, comes from a later age, when usage may have changed, but it illustrates the nature of the difficulties:[72]

> Excess Success (*diecheng* 跌成) is an old gambling game. In those bygone days people called it Windfall Gambling (*shibo* 拾博).[73] The variety that used 3 cash was Three Stars, the one that used 6 cash was Six Complete (*liu cheng* 六成), and the 8-cash version Eight Salutations (*ba cha* 八叉). When *all* were heads or *all* tails this was a Complete (*cheng* 成). Four heads and four tails was a Celestial Division (*Tian fen* 天分).... The game of Excess Success was called 'Pures' (*chun* 純) in ancient times. Under the Yuan dynasty, Li Wenwei wrote in his drama *Yan Qing bo yu*:... "If you want to gamble you will need to throw the Five Pure (*wuchun* 五純) and the Six Pure (*liu chun* 六純)." The Five Pure is what we today call 'One Short' (*yao yi* 幼一). The Six Pure is the Grand Complete (*da cheng* 大成).

The opacity of the terminology thus remains a problem, and so many of our textual interpretations are tentative. What is clear, though, is that the intensified commercialization and monetization of the Song economy were accompanied by an interweaving of gambling practices into routine market operations, and by a professionalization of gambling in a variety of ways.

To conclude, by the eve of modern times at the latest there were people in China who had a practical understanding of how the determination of probable frequencies could be carried out for the simplest gambling games, and this knowledge probably went back at least to Song times. We move on to practices where path-analysis would have opened up a probabilistic calculus with philosophical implications.

Divination

It is a commonplace that gambling and divination have much in common. Cheng Dachang tells the following story about the mid-sixth century CE:[74]

> According to the *History of the Northern Dynasties*, Xiao Chalu, the Lord of Liang, presented the Emperor Wen of the [Northern[75]] Zhou dynasty with an agate goblet. The latter took hold of it and, looking at his senior ministers, said: "If any of you can get an all-black with the casting-spills (*tou* 頭) used in *chupu* I will give him this goblet." Several of them had already failed to do so when the turn came to Xie Duan. He thereupon took hold of the casting-spills and spoke as follows: "I am not acting as I am because this goblet is to be prized but only because my thoughts may thus make manifest their sincerity." He tossed the five spills, and they were all of them black. Emperor Wen thereupon gave him the goblet.

The implication is clearly that Xie thought, or professed to think, that his sincerity could touch numinous powers of some sort. His success or failure were therefore not to be deemed to be due to vulgar luck. The similarity of the casting-spills to the yarrow-stalks used for selecting a hexagram when consulting the *Book of Changes* (discussed below) is obvious.

But how are we to interpret the story? A modern view might be that the monarch knew that he had a roughly even chance of keeping the cup, provided not more than 16 or so ministers were entitled to try their fortune with the spills, and that he was simply enjoying playing, partly as a gamble for his own entertainment, partly to do the ministers a favor, and partly no doubt to observe, in amused royal fashion, their cupidity. But there is another way of looking at it. He may have wanted to know who was inherently ‘lucky’. Consider the following anecdote from the *History of Qi*:[76]

> Li Anmin was once playing *chupu* with the Emperor Ming[77] and threw an all-black 5 times in succession. The Emperor was extremely startled, and declared: “Your face, Minister, is as square as a field!” He bestowed on him the rank of Marquis Prime Minister.[78] People said that his throwing the spills and obtaining a fat return was not a once-off lucky hit.

Squareness (*fang* 方) in a face was regarded as handsome, and the term also referred to high virtue. The Emperor must have had some idea of the vanishingly low probability of 5 all-blacks in a sequence ($(1/32)^5$ is 0·00000003, or three times in a hundred million.) Otherwise he would not have been so startled. A sense of what we might loosely call the ‘objective’ chances was needed to set in context the exceptional ‘luck’ of Li Anmin. In politics it is useful to have such people with personal luck on one’s side. Hence the imperial favor.

i. The coin oracle

The ancient Chinese scripture known in the West as *The Book of Changes*, and in Chinese most commonly as the *Yi* 易 [Changes] or *Zhou Yi* 周易 [Changes of the Zhou],[79] provided a potential practical laboratory for discovering the elementary laws of probability. Its surface mechanisms would have been familiar over the twenty centuries of imperial China to many hundreds of millions of Chinese in aggregate since the use of the book first became well established in the third century BCE.[80]

The *Changes* is a composite, many-layered text that has served both as a manual of prognostication and of wise counsel, but has been most revered for the philosophy found in its commentaries, and as a source of numerological speculation.[81] It may be loosely thought of as a sort of Chinese version of the medieval Jewish Kabbala. The numinous aura with which it was surrounded may also have become an obstacle to its dispassionate analysis.

The book is based on 64 hexagrams (*gua* 卦). In their classical form these are sets of 6 horizontal parallel lines constituting all the 2^6 possible distinct orderings of 2 types of line, which can be characterized as complete ‘—’ and broken ‘- -’, or as ‘hard’ and ‘soft’, or as ‘yang’ and ‘yin’. Thus the idea of a *permutation* is implicit in the evolved system, although the graphical origins of the hexagrams may have been sequences of archaic numerals with quite other numerological implications.[82] The underlying idea is that there is always a hexagram that either is, or symbolizes, any particular situation in the world as it currently applies to the questioner seeking foreknowledge and advice. It has, of course, to be correctly identified. The system is the abstract form of the cosmos as it moves through time.

The lines have a further property. They can be either changing (the so-called ‘old’ lines) or unchanging (the so-called ‘young’ lines). It was this innovation, associated with the Zhou dynasty, that may have given the *Changes* its name, in contrast with analogous earlier systems.[83] If the hexagram determined as the one relevant to the inquirer’s question contains some lines that are changing lines, the future development of the situation is regarded as symbolized by the transition from this hexagram to the one formed by altering the changing lines into their opposites: complete to broken and vice versa.

Old notation represented changing yang as ‘ø’ and changing yin as ‘—x—’. A variety of representations of hexagrams in modern notation are also possible. If yang is ‘positive’, and yin ‘negative’, youth ‘imaginary’ and old age ‘real’, we could for example set old yang = 1, young yin = *i*, old yin = -1, and young yang = -*i*, where *i* is the $\sqrt{-1}$. Representing inherent change would require multiplying each *changing* numeral in the string of 6 by *i*, so old yin becomes young yang, for example, and young yang old yang.[84] One justification for these assignations of values is that the traditional identification of the 4 types of line with the 4 seasons of the year matches up. Old yang is traditionally summer, young yin autumn, old yin winter, and young yang spring.[85] So, starting in summer, successive multiplications by *i* cycle us around the years: <1, *i*, - 1, - *i*, 1, etc>.

The simplest and fastest way of determining the applicable hexagram was by 6 tosses of 3 coins at time, each 3-coin toss determining a line, beginning at the bottom. As we have seen, traditional Chinese coins had an obverse and reverse, the equivalent of our ‘heads’ and ‘tails’. We may say that HHH yielded a *changing* complete line <1>, TTT a changing broken one <-1>, the 3 permutations of TTH an *unchanging* complete line <-*i*>, and the 3 of HHT an unchanging broken one <*i*>. In accordance with what is required by the traditional conception of the system, multiplying by *i* converts a static line to a changing one, and multiplying a changing line by *i* converts yang to yin and yin to yang.

It is hard for us today to imagine a Chinese scholar or fortune-teller *not* noticing that over the long run the percentages of hexagrams with given numbers of changing lines tended to converge to characteristic values. In round figures:

Number of changing lines	*Approximate percentage*	
0	18	
1	36	
2	30	
3	13	
4	3	
5		4 out of every 1,000
6		24 out of every 100,000

If — *if* — he had wanted to explain this pattern, he would have had to discover the following points:

i. The *probability* that a line determined by a toss of 3 coins will be a changing one is 1/4, and the probability that it will be unchanging is 3/4. The Chinese were accustomed to frequency counts: for example, the number of people out of every 10 in a county who died of an epidemic in a particular year.

There is, though, a subtle shift from a recorded frequency to a ‘probability’. This latter specifies either a general timeless case, or predicts a likelihood in the future. In one modern view, a probability is the limit of a frequency as the number of trials determining the frequency tends toward infinity. For the historian a more robust point of view is that a ‘frequency’ is an *observed* feature of a process in the ‘real’ world, and that a ‘probability’ is a defined or deduced feature of an *abstract* intellectual model, and that the greater or lesser convergence of the first with the second in a given case is, ultimately, no more than something that can only be established empirically. The usefulness for historical work of this way of looking at probability is that it focuses attention on the question of the capacity of a culture to create abstract models of a usable sort.

The specific conceptual difficulty here is seeing that there are 3 *possible paths* to each distinct mixed *combination* of H and T (one such being HHT, HTH, and THH giving <*i* >).

ii. Determining the probability that 2 successive 3-tosses will both yield an *identical* outcome requires the *multiplication* of the separate distinct probabilities. In the case of 2 successive unchanging lines this is 1/4 × 1/4 = 1/16. For this a mastery of the *multiplication of fractions* is needed, but this, as has been shown above, was no problem for the Chinese from late classical or early medieval times onward.[86] Examining experience over many trials also necessitates the practice of *recording results.* This was sometimes done, as in the 5-dice game Top The List discussed in a later section.

iii. Finding the probability of particular *sequences* defined *only* by their number of changing and unchanging lines requires multiplying each of the probabilities established by means of (i) and (ii) by the number of *distinct possible pathways* to the result. Thus there are 6 possible pathways to distinct hexagrams with only 1 changing line, namely by locating it in each of the 6 positions. Hence,with subscripts *r*, *n* indicating ‘changing’ or <real> and not changing or <imaginary> respectively:

$$p_{1,5} = (0{\cdot}25 \times 0{\cdot}75^5) \times 6 = 0{\cdot}355957031 \qquad (4).$$

For the small numbers concerned, the numbers of distinct pathways can be found by exhaustive tabulation and the elimination of duplicates. Doing this reveals that the distinct pathways to hexagrams with from 0 to 6 changing lines are: 1, 6, 15, 20, 15, 6, and 1.

If our imagined scholar or fortune-teller had been prompted by curiosity at this symmetry to look for a general rule he would have needed to discover two more points:

iv. The number of distinct ordered sequences for *n* distinct elements is ‘factorial’ *n*, defined earlier.

v. When some of the *n* elements are *not distinct*, the *n*! has to be divided by (*a*! × *b*! × ...) where *a*, *b*, ... are the numbers in each group of different non-distinct elements. For the case of hexagrams with 3 changing and 3 unchanging lines we have, for example, with ‘P’ denoting ‘paths’,

$$P_{3,\,3} = \frac{6!}{3! \times 3!} = 20 \qquad (5).$$

Since the number of permutations as defined in (iv) can, once more, be found by simple tabulation, its discovery presents no intrinsic difficulties, provided that one is interested in looking for it in the first place. Equations such as (5) are refinements, neat but not necessary.

The precise probabilities $p_{c,i}$ for coin-tossing-generated hexagrams with *i* changing lines are shown in Table 2.

TABLE 2

Probabilities of coin-generated hexagrams with given numbers of changing lines

$p_{c,\,0} = 0{\cdot}75^6$	$= 0{\cdot}177\ 978\ 516$
$p_{c,\,1} = (0{\cdot}25 \times 0{\cdot}75^5) \times 6$	$= 0{\cdot}355\ 957\ 031$
$p_{c,\,2} = (0{\cdot}25^2 \times 0{\cdot}75^4) \times 15$	$= 0{\cdot}296\ 630\ 859$
$p_{c,\,3} = (0{\cdot}25^3 \times 0{\cdot}75^3) \times 20$	$= 0{\cdot}131\ 835\ 938$
$p_{c,\,4} = (0{\cdot}25^4 \times 0{\cdot}75^{2)}) \times 15$	$= 0{\cdot}032\ 958\ 984$
$p_{c,\,5} = (0{\cdot}25^5 \times 0{\cdot}75) \times 6$	$= 0{\cdot}004\ 394\ 531$
$p_{c,\,6} = 0{\cdot}25^6$	$= 0{\cdot}000\ 244\ 141$
Total	$1{\cdot}000\ 000\ 000$

These calculations show that the probabilities of all possible outcomes *sum to unity*. Since premodern Chinese mathematicians were familiar with decimals and other fractions it is not unreasonable to say they had the manipulative skills necessary to handle such an idea. The problem would have been the concept of 'unity'. What does it *mean* in such a context?

Developing this toolkit on the basis of the changing lines in the *Changes* would have been sufficient to launch a theory of probability. *Nothing of the sort actually happened.* Since there were probably at least thousands of Chinese every day for over two thousand years tossing coins to determine a hexagram in the *Changes*, for the purposes, presumably, of seeking knowledge of the future and/or advice as to the best course of action, we have to ask, why not? A lack of adequate computational skills seems unlikely, and this seems to favor less clearly identifiable factors such as culturally conditioned attitudes. It was widely accepted that successful divination required a sincere and humble state of mind that was receptive to the subtle influences of the cosmos.[87] Not one that prodded and poked at the heart of the mysteries. Yet torrents of numbers were produced — above all by Shao Yong in the eleventh century — to depict the way in which the hexagrams ruled the evolution of the world.[88] Calculation was as such acceptable.

In Former Han times under the early empire, moreover, this reverence was by no means universal. Dongfang Shuo, who was well-known for his self-aggrandizing boasting and his sense of humor, "divided the divining-stalks, and set out in order the lines of a hexagram," while being watched by the emperor Wu and a number of experts in magical numerological predictions, in order to give himself some much-needed clairvoyance when he was playing a game in which participants had to guess what was hidden under an upturned bowl (*shefu* 射覆). His successful answer was derided by a court prankster with the remark that Shuo was demented, and "only was right by luck. He did not reach the actual numbers." He was obliged to prove himself again. [89] Nor was he alone in using the *Changes* like this.[90] Wang Chong, in the first century CE, even wrote of such efforts with mild approval:[91] "In ancient times... they did not yet have Dongfang Shuo and Yi Shaojun, who could divine [from hexagrams] the object hidden beneath an upturned bowl. Although this is but a minor art, it is likewise a technique of the sages."

ii. The yarrow-stalk oracle

The question is in fact more complicated. The most highly regarded method for determining the appropriate hexagram was not the tossing of triples of coins. It was the manipulation of the stalks of the common yarrow (*Achillea millefolia*)[92] according to an involved procedure.[93] As I shall show below, *the yarrow-stalk* *oracle did not give precisely the same frequencies as the coin oracle*. The divergence is in some cases large enough to be observationally apparent to any experienced practitioner. It is curious that these discrepancies did not prompt an inquiry into the reasons behind them.

Each of the 6 lines was determined one at a time by a 3-step procedure that left a certain number of stalks in 3 groups in the hand of the diviner. These triples of numbers determined the nature of the line as shown in Table 3.[94]

TABLE 3

The lines determined by the triples of stalks remaining in the hand of the diviner

	Stalks			Modern symbol	Traditional name
Step	**1**	**2**	**3**		
	5	4	4	+1	Old Yang
	9	8	8	- 1	Old Yin
	9	8	4	- *i*	Young Yang
	5	8	8	- *i*	Young Yang
	9	4	8	- *i*	Young Yang
	9	4	4	+*i*	Young Yin
	5	4	8	+*i*	Young Yin
	5	8	4	+*i*	Young Yin

Finding the probabilities of these triples, such as (5,4,4) = <+1>, or (9,8,8) = <- 1>, requires a description of the essential parts of the divining procedure constituting the 3 steps in Table 3. Since this is tedious, it has been relegated to Appendix B. Table 4 shows the probabilities for the step entries in Table 2.

TABLE 4

The probabilities for the components determing the lines as shown in Table 3

	Stalks			Symbol	Probability
Step	**1**	**2**	**3**		
	36/47	22/42	20/38	+1	0·211 166 213
	11/47	18/38	14/30	- 1	0·051 735 722
	11/47	18/38	16/30	- *i*	0·059 126 540
	36/47	20/42	16/34	- *i*	0·171 643 125
	11/47	20/38	16/34	- *i*	0·057 967 196
	11/47	20/38	18/34	+ *i*	0·065 213 095
	36/47	22/42	18/38	+ *i*	0·190 049 592
	36/47	20/42	18/34	+ *i*	0·193 098 516
Yang	(rows 1, 3, 4, 5)				0·499 903 074
Yin	(rows 2, 6, 7, 8)				0·500 096 925
Total					0·999 999 999

From Table 4 it thus seems likely that when the changing stalks were introduced into divination the yarrow-stalk system was *engineered*, either with analytical *understanding* or more probably by empirical *experiment*, to produce an equal number of yang and yin outcomes. The discrepancy in the two probabilities in the yang and yin rows is only just over 19 hundred-thousandths. The structure in Table 4 is far from intuitively self-evident. Twelve separate components go to make up each of the yang and the yin probabilities. In other words, it could hardly have been set up on simple first principles to give this result.

An unescapable conclusion follows: either there was a substantial understanding of probability, which was kept *secret*, among those who created and refined the system,[95] or there was, at least in private and during a certain period, *a minimally reverential experimental attitude* to its mechanism, which was adjusted till it gave the best observed results. I favor the second alternative, but the first cannot be ruled out.

The consequence of giving primacy to yang/yin overall parity was *disequilibrium* in other subsidiary respects. The yarrow-stalk system was more than 4 times more likely to derive a changing yang <+1> than a changing yin <-1>. On the contrary, a young yin (the sum of the three <+*i* >s) with a probability of 0·448361203 was more than one-and-a-half times more likely than a young yang (the sum of the three <-*i*>s) at 0·288736861. What logic decreed that the universe should be so lopsided? The tenor of the analyses of Shao Yong and other *Changes* numerologists constantly implies perfect balance and parity between the two complementary cosmic forces. There a rotten mathematical core at the center of the philosophical apple, either unrecognized or perhaps unackowledged.

The probability of deriving a changing line, that is $p<+1> + p<-1>$, was 0·262901935 as opposed to the 0·25 when using the coin-tossing method. Unlike the two preceding examples, this last might not have been immediately obvious even to a regular practitioner, but some experienced gamblers are reputed to be able to notice differences of this order.

Table 4 allows one to calculate the odds of getting any particular hexagram. Thus the probability of obtaining #18 in R. Wilhelm's version of the *Changes*, gu_6 = 'Work on what has been spoiled', with a changing line at the top, that is of <+*i*, -*i*, -*i*, +*i*, +*i*, +1>, is $p<+i>^3 \times p<-i>^2 \times p<+1> = 0{\cdot}001\,586\,766$ or 16 times in every 10,000 yarrow-stalk divinations. The transform symbolizing the future trend is <+*i*, -*i*, -*i*, +*i*, +*i*, +*i*> or a static #46, *sheng* = 'Pushing upward', glossed as success associated with effort.

A final observation: it would have been possible, if not easy, for a practitioner with a modicum of conjuring skill, and the effects of the different numbers in the two heaps at what we would see as 'random division' time securely memorized,[96] to have *cheated* at the three critical moments at the beginning of each of the three steps when the pile was being divided. All he would have had to do is to add or remove one or two stalks to, or from, the smaller set. Presumably the reason the number of stalks used was as large as it was (namely, 49) was to make this sort of action harder.

Gambling

Another source of repetitive numerical patterns was gambling, and gambling was a Chinese passion from early times.[97] Many games depended on insight, psychology, and skill in addition to luck, but a few could be played more effectively on the foundation of relatively simple probabilistic analysis. A sample of this last group are analyzed later in this section.

The gap between divination and gambling was blurred. For some gamblers, gambling may well have been experienced as an attempt to make contact with hidden numinous powers, and to persuade them to bestow their favors. The same Chinese word, *bu* 卜, could be used to cover both practices. This emerges from the preface to a work written by the Tang official Fang Qianli called, presumably in irony, *Tossing Dice to Choose People's Status*:[98]

> In the spring of 838 I came north from the sea by [river-]boat, and made a stop on the north shore of Dongting Lake. The wind was extremely fierce, and we moored the vessel in a creek in the open country for three days. We met with two or three gentlemen who held the rank of scholars deemed fit to be presented to the Emperor. They entertained themselves with pairs of pairs of hollow dry bones [dice]. They took turns tossing these onto a board, and let whether the score was high or low determine how far their status advanced, and the different grades of office. They thus determined wealth and honor, and assigned poverty and mean status. Their game featured official positions from captain upward, and honors such as being prime minister or commander-in-chief, as well as containing such phenomena as repeatedly acquiring a fine reputation but thereafter making no

impact, and having the most modest beginnings but rising up as swiftly as a flame to an elevated rank. Broadly speaking, *success and fortune were not due to fine or vile personal qualities, but merely to the good or ill luck of the divining/gambling process* (*bu*). I would note that this game is what we now know as the 'Board-Game for Rising in the Bureaucracy'.

The word translated 'board-game', more accurately 'game-board', is *tu* 圖, which has the basic meanings of 'map', or 'chart', or 'to plan'. My feeling is that it refers here to the layout of the board on which the game was played. Modern Western versions of this concept have been popular in the recent past, an example being 'Ratrace' that features the contemporary business world.

It is important, though, not to interpret Chinese games in any one-dimensional way. In the imperial palace in the second century BCE a game of chess (*go*) played on the fourth day of the eighth lunar month was believed to confer good fortune for the following year on the victor, and a twelvemonth of sickness on the loser, unless he made an offering of silk thread to the Northern Dipper, a constellation central to divination.[99] The game had magical overtones. "Place a tooth shed by a dragon close to the board," counselled *The Secrets of Encircling Chess*, "and clever tricks will occur to you of their own accord in lateral fashion."[100] So, too, was chess a cherished pastime of the immortals.[101] "If human beings could count all the stars in heaven, then they would understand the position-powers of the pieces in encircling chess."[102] But it had other symbolic resonances. Here are a few lines from Ma Rong's *Rhapsody on Encircling Chess* written in the Later Han dynasty:[103]

Encircling chess, in a general way, is like handling soldiers in war:
The field of battle is represented by a three-by-three-foot board.
They entrench themselves, first, on the Four Roads, guard corners, rely on edges,
And protect their forces along the frontiers, watching over in all directions
Their lines like skeins of flying geese, rows of seamless intersections.

Overstepping restraint, they secure a foothold, attempt forays into the center,
Removing warriors taken prisoner so they cannot rejoin together,
And if those that are faced with destruction are spared, they succor them from the threat.

Both sides interlock in chaotic confusion, transgressing each other's territory,
As precipitate breakthroughs follow weaknesses opening in their defences.
They penetrate deeply, greedy for space, and slaughter the enemy's ranks,
In desperate frenzy rescuing friends as their space-holdings swell, and then vanish.

Compare this with the Later Han historian Ban Gu's image of the same game as a microcosmos:[104]

The board is invariably square, being the image of the principle of the Earth. Its paths are straight, being the virtuous essence of the Spirits Bright.[105] The game is for Black and White, being divided into the Dark Force and Bright Force. When the two arrays of counters are spread out, they resemble the constellations in the Heavens. Overall it is [an exemplar of] Royal Governance. In the empty spaces they establish precautionary dispositions for self-defence. They raise barriers on all sides to block any breach in what is filled up. If a single hole is overlooked, then ruin is irrecoverable.

Per contra, Pi Rixiu, an official and scholar of Tang times, regarded the game as morally evil:[106]

If one does not inflict hurt, one is oneself vanquished. If one does not deceive, one loses. If one does not struggle, one perishes. If one does not contrive false pretences, one is oneself thrown into confusion. This is inevitable where encircling chess is concerned. Even if grandmaster Yiqiu were to appear in the world again, ought we to pay him any attention?

How could sage-emperor Yao,[107] with his empathetic concern for others, and his devotion to public duty, his adherence to ritually appropriate behaviour, and his wisdom, have taught his son to have a heart that was bent on hurt and deceit, and the knowledge of how to contend and contrive?

> It seems certain that encircling chess first appeared during the period of the Warring States, with its plethora of military strategists.

But Huan Tan, a scholar and official of the Later Han, looked on chess in a dispassionate fashion as a test of skill:[108]

> The most skilled prevail by establishing themselves far and wide, so obtaining the greater part of the space. Those of an intermediate level of skill concentrate on the mutual protection of their pieces, so as to strive for the advantage. The least skilled guard their frontiers, and link together intersections in the corners.

Games of chance and skill were thus used as metaphors for life, but in too many different ways to permit simple generalization, and sometimes just seen as themselves, without deeper meaning.

Among the ordinary people the irrational religion of luck exerted an almost irresistible oppression. 'Colored-Cards Meetings' (*huahui* 花會) are an example.[109] They began in Fujian in the Song dynasty, and spread into other areas during the Qing. The game may have begun as a *sweepstake* within a group, with one player's choice excluding others from making that choice, a common amount of stake-money for each, and a payout nearly equal to the total stake-money from all the players. The *Record of Things Close Enough to Hear*, a miscellany compiled in the second quarter of the nineteenth century, tells a story about these meetings in the Chaozhou area in eastern Guangdong:[110]

> The Yao family in Raoping...had a wife whose face was like a flower in the springtime of her youth. She was elegantly stylish to an exceptional degree, and she and her husband were devoted to each other. She had had one son when her husband died.
>
> It so happened that in Fujian province there were places where Colored-Card meetings were held. In the Song dynasty, people had whistled 36 persons together and every day posted up a name [of one of 36 legendary or historical heroes] as the winning selection (*ri biao yi ming* 日標一名). In general the standard payout was 30 times the stake. These meetings had spread from Fujian to Raoping, and when she heard about them her face lit up with desire and the hope that she would win heavily. In not many months there was nothing left of the family property.
>
> She immediately thought of the Du family across the way. They had more than a thousand ounces of silver, and were of high social status. The best thing to do was to borrow her stake-money for the following day from them. She at once paid a visit to their mansion and said: "Lend me 50 ounces of silver, and if I win at the meeting I'll return it to you with interest. If I lose, then my own person and that of my son will become yours." The Du knew from everyday experience that she was of good moral character, and were greedy for her sexual charms. Mischief-makers also egged them on from the side. She signed the bond and left.
>
> After returning home that night she went to her husband's grave, where she wept and prayed to him: "The Colored-Card meetings have ruined us. No funds remain for food or clothing. If you have any magical powers, you may perchance help me in my dreams [with a tip for the winning name]. If not, then tomorrow your wife and son will both belong to others." When her prayer had finished she had the impression that her husband was saying to her in a dream: "I know your heart and mind [are true]. At midday tomorrow I shall help you from the world of shadows. You will be able entirely to clear off the silver you have borrowed. When you gamble with it, you will win. But not long afterwards you will be unable to avoid being charged before the magistrate."
>
> When morning had dawned, she did as he had told her, and indeed won a large sum. She went home and repaid double the amount she had borrowed from the Du family, but they raised an uproar, saying: "Previously this was betrothal money. It most certainly was not a loan!" They had every intention of forcing her to remarry, and so denounced her to the magistrate.
>
> The official governing Raoping at this time inquired into the story from beginning to end, and asked the wife what her intentions had been. "I only wished to maintain my fidelity as a widow," she answered. [She needed something to live on.] He then judged the case as follows:
>
> "To lend silver and take it back with interest that doubles the amount, while using the loan as a pretext for forcibly seizing someone else's wife is so blatantly evil[111] that it is hard for the law

to be indulgent."

He had the Du's severely flogged and displayed with punitive collars round their necks, confiscated their silver, and had the bond returned to the widowed wife. A severe prohibition was placed on Colored-Card meetings, to spare the people this scourge, and he further instructed her: "What is good for you will be to stay quietly at home. Intruding into the gambling arena in such an improper fashion on this occasion was not appropriate. Bearing in mind that your desire was to maintain your fidelity to your deceased husband, your heart is still to be commended." He excused her from the application of the law, and gave her words of encouragement.

Such belief in the possible intervention of the spirit world made the study of probability irrelevant. The distaste for gambling of this type among the educated class must also have inhibited many of them from examining it and also from writing about it in other than general and disapproving tones.

The casino-and-client variety that had developed by the end of the Qing was run on somewhat more complex lines by professional gambling operators. The name of one of 36 famous personages was first secretly written on a cloth that was then tightly wound on a roller (*tong* 筒) which was hung up horizontally on a frame where it was visible to all. Each participant picked the name(s) of one or more of these personages, usually on cards marked with various symbols, such as distinctive animals, to aid identification (hence the name given to these meetings), marked down his or her selection, attached some mark of personal identification, and put it into a closed chest with the stake-money attached. This stake could be of almost any size, large or small. The roller was subsequently unrolled at a time announced beforehand, and the winning name was revealed. The winners were those who had correctly guessed the famous personage whose name had been previously selected. The payoff was usually 30 times the stake, though this multiplier was sometimes dropped to 28, as in Shanghai. Customarily only 32 of the names of personages were actually used. Thus the much-revered Lin Yinjie, whose name was on the list in Shanghai and who was the patron deity there of colored-card gambling operations, is said never to have been used. It is not clear if his name could even so have been occasionally written on the roller-cloth, which would of course have ensured a total win for the operators. It seems unlikely, given probable public reaction. At some point an innovation was introduced in the presentation: the roller was shielded by a curtain on a stage that bore the previous winning name, and was raised when the name of the current winning personage, written on the cloth tightly wound on the roller behind, was to be, slowly, revealed to the gamblers. Slowly, so as to intensify the excitement.

The logical structure of the game resembled a simplified form of roulette, without such complexities as split bets, and bets on odd or even, or red or black, and high or low. The arrival of western roulette in Shanghai in the early twentieth century is in fact said to have significantly reduced the popularity of colored-cards meetings. There were various possible extra twists in the Chinese game, and some regional variations, but in the simplest case, if x stands for the amount of money staked, then the expected return r for a single bet on a single name would have been simply the probability of success times the payoff, or

$$\mathrm{EXP}(r) = \frac{1}{32} \times 30x = 0{\cdot}9375x \qquad (6).$$

In other words, assuming his stake was not returned to him in addition when he won,[112] a player would have expected at each round to lose a proportion of his stake equivalent to (1- 0·9375) or 0·0625, which is one-sixteenth, or slightly greater than the house take in United States casino roulette.[113] A guaranteed disaster over the long run. The Shanghai gamblers were clearly either unaware of such calculations, or indifferent to them. The case of the gambling operators is more problematic. As equation (6) shows, they did not need to rig the results in order to win, but there is evidence, given below, to suggest that they did rig them nonetheless. We have therefore to accept that they may have operated on the basis of convention or tradition, without any deeper understanding, and altered the odds (or rigged the outcome) when they suspected they could get away with extracting increased profits.

Here is part of Chen Dingshan's description of colored-cards meetings in Shanghai at the end of the empire and under the early republic.[114] At the time about which he was writing, the payoff had been dropped to 28 times the stake, so the r expected was $(28/32)x = 0{\cdot}875x$, or a 12·5 per cent take for the operators. Interestingly this is close to the level of the basic house take for the two-wheel 'double roulette', introduced at Monte Carlo in 1936, which rapidly proved unacceptable to European gamblers.[115] Not so in China.

> The 'seafarers' [*hangchuan* 航船 — itinerant touters for business] were divided into men's and women's groups. The male 'seafarers' specialized in going around business premises and enticing the staff and apprentices into coming gambling. The females concentrated on entering the mansions of rich and important people, where they lured the women from these respectable households into having a flutter. The rule was that, irrespective of whether the clients introduced won or not, the bosses would give the 'seafarers' 10 per cent of what they betted.[116] For this reason the unemployed and vagrants, and gadabout women, all worked part-time as 'seafarers'. There were more of them than ill-omened crows, relying on their cawing voices to tell people how 'the flowers were falling from heaven'.
>
> Thinking that, with payouts at 28 times the stake, vast profits were to be had here, everyone lost their moral willpower. Once greedy thoughts had surged up in them, people even bankrupted their families or lost their lives. Women who attended the colored-cards meetings went out at night in ever increasing numbers to wastelands, and hills with tombs on them, in order to pray for dreams with omens of future events. If they caught sight of a tiger in their dreams they would place a bet on Wang Kunshan. If they glimpsed a small Buddhist monk, they would wager on Fang Maolin. The Grotto of the Immortals in the Temple of the Jade Buddha at Jiangwan...[and several other shrines]...were all targets for those praying for such dreams. When men and women stayed out late at night or traveled after dark, they would come back intermingling with each other, and night-prowlers would track them and insult them.... Some there were, too, who thieved from graves and dug up the bones, bundling them together and taking them home. Here they would burn incense and make libations in order to pray for omens of what was to come. They would seem as if they were drunk or demented all day long, wasting their time and losing their jobs. It was a disaster worse than opium.
>
> Strangest of all were the 'civilian' and 'military' flag-brandishers. The civilians would make 36 paper flags, each with the name of one of the personages of the colored-cards meetings written on it, and, in the depths of the night, form groups to go out to derelict tombs and ancient graves where they would erect altars for incense-burning. They would stick the flags all round the grave-site, and once their praying was completed, sit to observe the direction of the wind. If the wind blew a flag over they would gather their joint resources the following day to place a huge bet that the 'gate' of this one of the personages would 'open'.[117]
>
> The military variety would look for a beggar, and clothe him in paper finery. Then they would take him to an old tomb where they would have a feast with wine, and kneel in a circle round him. When these obeisances were over, a spirit would possess the mendicant, and, through his mouth spit forth the name of a personage featured in the colored-cards meeting.
>
> There were also people who dressed themselves up as demon kings. They were led along by someone burning paper money and followed behind by two others blowing on trumpets. The would wind their way around the graves and through the suburbs, and make as if they were something [supernatural] that had been sighted. The next day rumors would be passed along through the streets that such-and-such a 'gate' would open. The 'seafarers' and 'roller watchers' (*tingtong* 聽筒) would spread gossip about everywhere and huge bets would flood in to the colored-cards meetings. But success for the participants would remain as elusive as before.
>
> It was not only the lower levels of society that were thus poisoned. Even the women of wealthy families every last one of them flocked to it like ducks to water. When they had pawned all they had, and lost their standing in the eyes of others, they would hang themselves from a beam or swallow a deadly drug....

The system could be rigged if need be. There was, as it happened, more than one cloth with a name on it wound around the roller, and if the operators, who kept a careful check on how much was being laid on which names, saw they were likely to lose heavily, it was simple enough for them, given the shielding curtain, to arrange to lower one with a different name from the one originally intended. This was termed 'opening an empty gate'. Such sleight-of-hand would usually spare the operators any net loss, since, as

Chen slightly quaintly put it, “it is extremely difficult for winning bets to be placed on all 32 gates.”

There is no evidence for a rudimentary sense of probability here, even on the part of the operators. They were apparently not willing to trust what must have been familiar trends in the frequencies, in a game where the odds were already in their favor. On the contrary, they fended off threatened heavy losses through last-minute cheating based on conjuring skills, rather than being reconciled to them as transient. And perhaps failed to see that the occasional massive payout was an incentive to customers to come back.

i. Morra, or ‘guess-fingers’

The game of morra had several names in Chinese. One was *huaqian* 豁拳, which literally means ‘opening the fist’; another was *caiquan* 猜拳, or ‘guess fist’; and a third was *muzhan* 拇戰, ‘thumb-war’. The rules are defined by the *Sea of Phrases* encyclopedia as follows: “Two people face each other and put out their fists with some of the fingers extended, while each of them guesses the total number of fingers so extended and announces this out loud.”[118] If both announcements are wrong or both are right, the result is a draw, and there is no score. If only one of the two players announces the right total he scores a number of points equal to the total number of fingers extended. Zeroes are allowed in some variants.[119] Morra can also be played by more than 2 people.

The game involves two quite distinct types of skill. One requires probabilistic thinking to determine the best proportions of each of the particular moves, such as ‘Show 2, say 3’, and so on. The other requires, first, operating as far as possible without any discernible pattern to one’s own moves that would allow one’s opponent to predict one’s next move at better odds than mere chance; and, second, detecting any pattern in one’s opponent’s moves that can be used to help predict what he will do next. Just using the best theoretical proportions of particular moves is of little use if they are played in a regular and hence predictable fashion. There is a need for randomization within predetermined frequency constraints.

Xie Zhaozhe refers to guess-fingers in his *Fivefold Miscellany* of 1608 in a passage that suggests that there was some understanding of both of these aspects in late Ming China:[120]

> Though guess-fingers is exceedingly vulgar, there are, even so, some who have a refined command of its techniques. Among the writings of Mr Yuan of Wumen is a *Thumb Classic*, and by relying on it he has been without equal in the world. So far I have not set eyes on it.
>
> At the Half Moon Spring in Deqing there is a Buddhist practitioner who has had a hundred-per-cent success at this game. People suspect him of using improper means, but none such exist. It is simply that he has an outstanding memory. He can remember more than ten plays [just] made by another person, and thereby intuits the directions in which the other shifts. The *Youyang Miscellany* [a Tang-dynasty work] calls this “observing the forms and noting the colors.” He obtains [the answer] in the way one detects a thief by discriminating perception.

The *Thumb Classic* more likely than not contained numerical recipes for the frequencies of particular moves. The Buddhist excelled at the detection of patterns that were characteristic of an opponent’s way of thinking.

What might these recipes have been in the book that Xie never saw? A theoretical discussion of morra, based on a modern analysis by Epstein,[121] may give some idea. Epstein considers a simplified version of the game, in which only 3 fingers are used by the 2 opponents, and the players have to guess only the number of fingers shown by their opponent.. This reduces the possible moves each side can make to 9, and so the payoff matrix has a 9 by 9 antisymmetric form[122] since A’s gain is B’s loss and vice versa, as shown in Table 5, where what each player says is in bold type. To illustrate: if A shows 2 and says **3** (row 6), and B shows 2 and says **2** (column 5), the payoff is -4. Minus, because B wins, correctly predicting the number of fingers shown by A; and 4, because this is the total of fingers extended by both.

The question to be solved is: With what frequencies should A use each of the 9 possible show-and-say combinations available to him, taking care to avoid predictable patterns? The answer will the the same for B because the game is symmetric. A general approach to such issues has been developed in the twentieth-century *theory of games*, but it would be anachronistic to imagine that Mr Yuan of Wumen could have worked this out more than three hundred years earlier. There is, though, a plausible solution based on the inspection of the payoff matrix as shown in Table 5, and some simple logical explorations. The numerals in italics label the moves at each player's disposal. **A** and **B** label the 2 players.

It is apparent that moves *3*, *5*, and *7* together form a 0-sum block, in the sense that if *both* players play *only* these 3 moves there is no score, and therefore a draw results. There are no other such 3 × 3 blocks. If we now look, from **A**'s point of view, at the payoffs owing at the intersections of moves *3*, *5*, and *7* and the other remaining 6 moves, it is apparent that he can able to adjust the frequencies with which he plays the *3*, *5*, *7* group so that overall, provided he is otherwise unpredictable, he wins a plus score if **B** departs from *3*, *5*, and *7*. This non-0 payoff matrix is shown in Table 6, with the residual payoffs in the final column. At equal frequencies the payoffs balance, but if *3* can be given a larger weight than *5*, and *5* a larger weight than *7*, then **A** will defeat **B** if the latter does not adhere to the *3*, *5*, *7* strategy.

TABLE 5

Payoff matrix for 3-finger morra (after Epstein)

		1	*2*	***3***	*4*	***5***	*6*	***7***	*8*	*9*
	B:	1,**1**	1,**2**	1,**3**	2,**1**	2,**2**	2,**3**	3,**1**	3,**2**	3,**3**
A:										
1	1,**1**	0	2	2	-3	0	0	-4	0	0
2	1,**2**	-2	0	0	0	3	3	-4	0	0
3	1,**3**	-2	0	0	-3	0	0	0	4	4
4	2,**1**	3	0	3	0	-4	0	0	-5	0
5	2,**2**	0	-3	0	4	0	4	0	-5	0
6	2,**3**	0	-3	0	0	-4	0	5	0	5
7	3,**1**	4	4	0	0	0	-5	0	0	-6
8	3,**2**	0	0	-4	5	5	0	0	0	-6
9	3,**3**	0	0	-4	0	0	-5	6	6	0

TABLE 6

Residual payoffs in 3-finger morra after the removal of the 0-sum block

	1	*2*	*4*	*6*	*8*	*9*	*Residual score at unweighted frequencies*
3	-2	0	-3	0	4	4	+3
5	0	-3	4	4	-5	0	0
7	4	4	0	-5	0	-6	- 3

The simplest set of weights for (*3*, *5*, *7*) is (5/12, 4/12, 3/12).[123] The payoffs for **A** from the combinations shown in Table 6 now become (+15/12, 0/12, -9/12), a plus score over the long run of 6/12 per play if **B** avoids *3*, *5*, and *7*. Something comparable to the above, but for 4 or 5 fingers, is likely to have been what was in Mr Yuan's *Thumb Classic*.

For the simplified game given above, one way to pseudo-randomize the distribution by twelfths without giving away any secrets, as by visibly tossing a die or dice, would be to use the 60-cycle of pairs of the 10 'celestial trunks' (*Tian gan* 天干) and 12 'earthly branches' (*di zhi* 地支) used both for days and years in imperial times, and known by heart by every educated Chinese.[124] This cycle runs T_1B_1, T_2B_2, ... $T_{10}B_{10}$, T_1B_{11}, T_2B_{12}, T_3B_1, T_4B_2, etc, where 'T' is 'trunks' and 'B' is 'branches'. The player could progress mentally through the 60-step cycle by one step at each round, and assign distinctive, non-sequential, sets of 5, 4, and 3 'branches' to activate moves *3*, *5*, and *7* respectively. For example, for move *3* in Tables 5 and 6, which should have frequency 5/12, start with the the third branch, *yin* 寅, and pick *chen* 辰, *wu* 午, *wei* 未, and *you* 酉in addition. This generates 5 cycles spaced at gaps of (2, 2, 1, 2, 5), the intervals being filled with *5* and *7* as determined by other T+B groups. For defeating Buddhist practitioners with computer-like memories, deeper numerical disguise might have been necessary.

This line of thinking is speculation, but it is constrained by the need to show how Mr Yuan could conceivably both have been regarded as being "without equal in the world," without anticipating the modern theory of games. As with the analysis of the divination techniques of the *Book of Changes* given earlier, the indications are that there must have been a greater knowledge of probability in premodern China than appears in the surviving texts.

ii. Dice games

The conventional birth-date of the probability calculus in the West is the discussion between Fermat and Pascal in 1654 on how to share out the stake-money appropriately when a gambling game was stopped before it was concluded.[125]` Fermat gave the following answer to a sample problem: if 3 players engaged in a gambling competition stop when A still needs 1 game to win, and B and C both need 2 games, and each has an equal 1/3 chance of winning any given game, how is the money they have wagered to be fairly divided? A can win in 1, 2, or 3 games. There is 1 path for the first of these possibilities, with a probability of 1/3. For the second possibility, there are 2 paths: either B or C wins the first game, and A wins the second. The probability of this is $2 \times (1/3)^2$. For the third possibility, B wins the first game and C the second, or *vice versa*, and A wins the third. The probability of this happening is $2 \times (1/3)^3$. The sum of the 3 ways for A to triumph is thus 9/27 + 6/27 + 2/27 = 17/27. So A should get 17/27 of the stake-money, and B and C share the remainder with 5/27 each. The key features of this calculus are using the multiplication of probabilities for 'and', and their addition for 'or', taking all possible paths into account, and allocating its own distinctive probability to each path comprising a distinct order of events.

Gambling games existed in mid- and late-imperial China that could have provided, when simplified, models for this sort of thinking. One such was the Qing-dynasty game Top The List, *Zhuangyuan chou* 狀元籌, in other words gaining dicing-points to secure an imaginary first place in the Palace Examinations.[126] It was played mainly at the New Year by an indeterminate number of players each in turn tossing 6 dice, with a record being kept of their scores in a notebook. The first player to reach (or, presumably, surpass) an aggregate total of 64 was the winner. There were also intermediate scores, such as 32 for second or third place in the exams, that presumably carried some lesser reward. Imagine, however, we follow Shen Gua's example and look at a simplified version, with only 1 die and 2 players, in which the winning total is 16. Suppose that A has 12 points and B 10 when the game is forced to stop. How much should they each get of the money jointly wagered? The details are conceptually simple but computationally tedious. They have therefore been put in Appendix C. Overall, it can be determined by separately summing the contributions of the decided games won by A and B at each move that A has an 87% probability of winning and B 13%. The stake-money should be divided accordingly.

This calculation would have been technically elementary for late-imperial Chinese mathematicians, though they would have needed patience and persistence for the bookkeeping. What the demonstration highlights is that, in the Chinese case, where these calculations were *not* undertaken, or at least never made public, the key issue, from our perspective, was that no one asked the heuristically crucial *question*: How does one compare the chances of players with different scores of winning in an uncompleted game?

The reader may object at this point: "The discussion has assumed that Chinese thinkers of the imperial period *could* have solved the problem of creating a basic calculus of probability simply by elementary *tabulations*. Granted in a technical sense, but is there any evidence that they *actually* used tabulations in the context of games of chance?" The answer is 'yes'. As early as 1134 we find the woman scholar, poet, and devotee of board-games Li Qingzhao listing all the 56 possible patterns (*cai* 采) for the 3 dice needed in her new game Horses Out! (*da ma* 打馬).[127] This was a race game, with hazards and mutual interference on the part of the players, that had distant links with backgammon, though unlike bakgammon everyone moved round the track in the same direction, which meant that participation was not limited to two people. The ordering in her table and the 3-dice pip-scores involved were as follows:

1. Patterns earning a bonus (*shangcai* 賞采)

1.1 The six triples, with (4, 4, 4) in first place, followed by (6, 6, 6) and the others in descending size.

1.2 Twenty-one other patterns: 17, 15, 14, 5 × 13, 2 × 12, 10, 4 × 9, 3 × 8, **11, 2 × 10**.

2. Patterns carrying a penalty (*facai* 罰采)
2 patterns: 6, 4.

3. Neutral patterns
27 patterns: 2 × 16, 15, 14, 3 × 12, 5 × 11, 10, 2 × 8, 4 × 7, 2 × 5, 6, **2 × 14, 2 × 10,** 9

Ordering was thus in the first place by the *categories* of 'bonus', 'penalty' and 'neutral'. For example, *xiehuan* 靴桓 'boot-wax tree [?]' was the only bonus-bearing 11, and so put in a different category from the other 11s. Ordering by *pip-totals* was secondary, with inconsistencies in the final block (shown in bold). Each combination also carried *a name*. Thus (5, 3, 5) was *zaohe* 皂鶴 'black crane'.

The purpose of having this tabulation was so the players could refer to it in case of doubt about the allocation of rewards or penalties. It distinguished clearly between the various patterns that scored the same total. Thus 11 was scored by (2, 3, 6), (3, 5, 3), (4, 3, 4), (2, 4, 5), (5, 1, 5), and (4, 1, 6). These six all had different names.

The names had fragmentary elements of systematization. Thus, highlighting the common numerical pattern in bold, the names for (**2**, 5, **2**), (**2**, 4, **2**), (**2**, 3, **2**), and (**2**, 6, **2**) all contain the character *jia* 夾. This means 'press, squeeze, or pick up with pincers' as well as 'double, lined' and 'carry secretly'. Not every pattern of the form (**2**, *n*, **2**) was a *jia*, however. Patchy taxonomy is shown by the designation of (**2**, 1, **2**) as *xiaozui* 小觜 'little Zui' (a constellation). Likewise (6, 1, 6) and (5, 1, 5) were 'big spear' and 'little spear' respectively, and (5, 3, 5) 'black crane', and (4, 3, 4) 'red crane'.

Other games had similar tables. An example is the dicing game of Red Fours Out! (*chuhong* 除紅), which was perfected in Yuan times, and required 4 dice.[128] The most important for the present discussion is Drunken Verdure (*zuilu* 醉綠), which is probably no later than late Ming and may be earlier.[129] This was a game with 'rewards' and 'penalties' in cups of wine. It used 5 dice. The original Chinese tabulation lists *all* 252 combinations in an *analytical* order. These combinations were composed of subunits that can be defined by a notation using '*n*S' and '*n*D' to indicate *n* similar or *n* different digits, with the prime, as in '*n*S´', to mark that the similar digits in a second group are not the same as those in the first 'S' group in the same combination. This historical tabulation is summarized in columns **A** and **B** of Table 7. Note how the entries in **B** are in ascending order of size. Column A´ provides illustrative examples added by me.

TABLE 7
Traditional Patterns and Modern Probabilities from the Dice-game Drunken Verdure

A *Patterns*	(A´) *(Example)*	**B** *Combinations*	C *Orderings per Combination*	D (= B × C) *Paths*	E (= D/ΣD) *Probabilities*
5S	(66666)	6	1	6	0·000 77
5D	(64321)	6	120	720	0·092 59
4S 1D	(66661)	30	5	150	0·019 29
3S 2S´	(66611)	30	10	300	0·038 58
3S 2D	(66612)	60	20	1200	0·154 32
2S 2S´ D	(66112)	60	30	1800	0·231 48
2S 3D	(66123)	60	60	3600	0·462 96
Totals		252		7776	0·999 99

Note: Results achieved in premodern China are in the two columns whose labels are in **bold type**.

The entries in columns C, D, and E are what we would now add. C gives the numbers of different *orderings of elements* within each combination, D the number of *paths* to a given type of combination, and E the *probability* of some one of this type of combination turning up. We can therefore be formal: assuming that in the future nothing of critical novelty in this regard turns up in the sources, *the failure of* *late-imperial Chinese society to develop probabilistic thinking is defined by the inability to move from column **B** to column C in Table 7.* Both D and E follow immediately from the addition of C. The row entries D_i are the products of the corresponding row entries $\mathbf{B_i} \times C_i$. $\sum D$, the sum of all the D_i , is 7776, and the row entries E_i are $D_i/\sum D$. Any explanation that does not provide for *both* the creation of B *and* the *non*-creation of *C* can be ruled *a priori* to be unsatisfactory. This is an unexpectedly tight constraint. What can be usefully said, at least by way of getting started?

First, C is trivially easy to find, provided one is looking for it. Thus, for example, to determine the fourth row, one scribbles down and double-checks something like '66611, 16661, 11666, 61166, 66116; 61661, 16166, 61616, 66161, 16616'. One does not have to use the formulae for each row, namely 5!/5!, 5!/1!, 5!/4!, 5!/(3! × 2!), 5!/3!, 5!/(2! × 2!), and 5!/2! though it is handy if one can. The blockage was not a technical mathematical one. It was *conceptual.*

Second, it was a slight overstatement above to say that the original table for game Drunken Verdure created column **B**. The numbers in each group are not explicitly stated, though the members of each type are all grouped together and can be counted in a matter of moments. Li Qingzhao had long ago counted the total figure for combinations in the 3-dice Horses Out! (the analogue of 252 in column **B**), but this is not quite the same as summing and listing the numbers of each specific type.

Third, the *names* are a probable clue. Only the 37 most important patterns are named in Drunken Verdure, in contrast to the 56 of Horses Out!, where they all are. But these important patterns have something akin to a personality and also a normative spatial structure. In the dot format used for the illustrations, the pattern 66661, 'Fair Sun in the Midst of Heaven' (*li ri zhong Tian* 麗日中天), is displayed as follows: a single dot, presumbly the sun, is centered above two dot-sixes, vertically aligned like a pair of double columns below it, with two further vertical dot-sixes below these, but set farther apart. It is easy to imagine players rearranging the dice that fell with these faces upward, whether on the table or in their minds, to make them take on their 'proper' pattern, regardless of how they had actually fallen. This normative structure may have inhibited the awareness that this pattern could form in 5 different ways.

In many respects the multi-dice patterns (*cai* 采) resembled the 6-line hexagrams (*gua* 卦) in the *Book of Changes*. Both were miniature pattern-principles (*li* 理), guiding templates of the universe. Most importantly, they shared the comparatively isolated, and minimally interactive'black-box' character of these two concepts. Dice-patterns, with their distinctive names, were perhaps perceived as evanescent expressions of permanent abstract pattern-principles or hexagram-equivalents in minature, and therefore had an imagined coherence of character that assorted ill with the notion that they were no more than the endpoints of multibranched aleatoric paths that could reach them by many routes.

Chance Events

What was the premodern Chinese concept of 'chance'? There is a suggestive image from pre-imperial times in *The Preface to Master Sun*:[130]

> A pearl is traveling inside a basin. Its motions, crosswise and aslant, and in curves and in straight lines, cannot be known in their entirety. What can always be known is that the pearl will not leave the basin.

This prompts one to wonder if any premodern Chinese thinkers explored the idea that there were cases where events that were individually unpredictable (whether because of human incapacity or their inherent character) gradually fell into ensembles whose structures and frequencies stabilized as the number of such events increased. If there were, examples have still to be found.

Let us conclude with some speculation on how the concept of 'chance' may have developed in China. In archaic times, more than three thousand years ago, the universe was seen as ruled by spirits. Divination by the cracks opened in a turtle's undershell by the application of a heated point was used to see which of them was at work in particular phenomena, and to foretell if there would be disasters (*huo* 禍) or harm (*hai* 害) or good luck (*ji* 吉). [131] In such a world there can be favorable and unfavorable fortune but, in a sense, no chance. By classical times, *per contra*, in the first millennium CE, the notion of a 'constancy' (*chang* 常) in the indissolubly interwoven workings of the human and natural worlds had crystallized. This was summed up in the familiar phrase 'the regular paths of Heaven and the dutiful public-spiritedness of Earth' (*Tian-jing Di-yi* 天經地義). By the third century BCE, the philosopher Xunzi had even so desacralized 'Heaven' that it was virtually synonymous with what we would call 'nature', and one of its essential characteristics a constant behavior.[132] The commonest view, from classical times to late-imperial times, was, however, a sort of compromise between these two positions: irregularities or abnormalities, such as droughts, in the otherwise constant behavior of nature were a sign of an adverse judgement on human behavior, usually by Heaven.[133] They therefore carried information. There were no *natural*, macroscopic, examples of *equiprobable* events, if we except the birth of approximately equal numbers of male and female children.

In the context of these remarks the *artificial* creation of equiprobability by means of technology — binary throwing-spills and dice, and the tossing of coins — can be appreciated for the peculiar and important event it was. In the simplest case, where 2 outcomes have an equal likelihood of occurring, the pattern of exceptional events occurring against a background of normal expectation vanishes. So does information. Half-and-half outcomes — the *zi* or *mu* of a spun coin, the black or white of the *chupu* spills, and the odd or even numbers of Yan Zhihui's guessing game — are in themselves meaningless.

It may have been thought that in such delicately balanced circumstances the numinous forces could more easily exert an effect on objects. Whether this was the case or not, meaning was reinserted into the void by assigning particular characteristics, often labelled with names, to given combinations and permutations. Thus, as regards combinations, the *black* faces showing in the 5 tossed *chupu* spills were given precedence over the white ones, and the numbers showing in a set of 5 were given the meaning of *scores*, rarer being higher. Explicitly, combinations with 5, 4, 3, 2, 1 or 0 black faces showing, which respectively turned up in frequencies of approximately 1, 5, 10, 10, 5, and 1 times out of 32 (these numbers from Pascal's triangle being countable by inspection or calculable from the numbers of possible permutations leading to particular proportions of black and white faces showing), determined — roughly speaking — the hierarchy of success. As for straighforward *permutations*, the $2^6 = 64$ ordered configurations that can be made from 2 distinct elements taken 6 at a time yielded the basic individually named 6-line hexagrams of the *Changes*. ('Basic' in this context means without the further distinctions created by changing or static lines.) If these static hexagrams had been selected by tossing 6 *chupu*-type spills one by one in sequence, there would have been a parallel use of combinations and permutations in two essentially identical systems. This didn't happen, of course, and more complex techniques were used in divination, but the thought suggests that the probable late-archaic shift from static to dynamic lines in the *Changes*, and in gambling the replacement of the spills by the cubical 6-faced die in medieval times, may have made thinking about combinations and permutations together harder than it would otherwise have been. Merely looking at one or other of the various tables of the hexagrams in common use, and a minute's counting, would have made it apparent that there were, for example 20 distinct ways to reach hexagrams that had 3 unbroken and 3 broken lines,[134] but only 1 to reach a hexagram with all its lines solid.[135] If one thinks of them as *chupu* scores, that is.... But they didn't, and the probabilities expressed as 20/64, 1/64, etc., never appeared.

There were exceptions to the 'compromise' alluded to above. Perhaps the most interesting of these was formulated by Wang Chong, who flourished in the first century CE, and wrote a massive work, the *Lunheng* 論衡, or *Discourses Weighed in the Balance*, characterized by tantalizing elements of what seem to us scepticism and materialism.[136] From a present-day perspective his style of thinking looks at first sight like one that could have developed into a more precise concern with probability. The nearest he comes to the explicit idea of equiprobability, though, is in a moralizing metaphor:

> If you toss a spherical object onto the ground, it can go east, west, south, or north, any of these being possible. If a walking-stick is knocked flat by a blow, it will give only a slight quiver before coming to a standstill. If a square-sided object is put on the ground, it will stop once it has been dropped, and not shift unless someone displaces it. The Confucian of high moral worth is like a square-sided object in the world. That he is hard to turn or shift is that this requires the agency of others.[137]

Stubbornness may thus sometimes be a virtue, but our fortunes are aleatoric:

> When ants run along the ground, people raise their feet and pass among them. Where these feet tread, the ants are pressed to death. Where these feet do not trample, the ants are unharmed and full of life.... When a spider knots her web and flying insects pass through it, *some* escape and *some* are caught.[138]

The word *huo* 或, translated here as 'some', also has the meanings of 'perhaps', 'perchance', and 'in some cases'. It is a component of one of the modern Chinese words for 'probability', *huolü* 或率, 'the rate of someness'.

Being unlucky was not necessarily morally deserved:

> If someone is standing beneath a high wall and is crushed when it disintegrates, or is striding along a fissured bank and perishes when it collapses, this is a fleeting encounter with something that has no initiating cause (*qing yu wuduan* 輕遇無端), and is therefore *bad luck* (*bu xing* 不幸).[139]

'Happenstance' (*zao.zhe* 遭者) was "encountering an abnormal turn of events" (*feichang zhi bian* 非常之變).[140] 'Predetermined fate' or *ming* 命, a term that often elsewhere meant 'command', nonetheless ruled the outcomes of life, and at some moments it was possible to detect it in advance determining the character of coming events:

> The younger brother of Empress Dou was called Guangguo. Their family was poor, and at the age of three or four he was seized by other people and sold. His family had no idea where he was. He was sold on from one family to another more than ten times. When he came to Yiyang [in modern He'nan], his master sent him into the hills to make charcoal. One evening, when it was cold, more than a hundred men lay down at the foot of the [smoldering] charcoal mound. When it then subsided onto them, they were all crushed to death, Guangguo being the only one who escaped. *He had himself divined that in a few days time he would be given a portent.*
>
> Subsequently he made his home in Chang'an, where he heard that Empress Dou had recently ascended to the throne...so he submitted a report giving an account of himself. Empress Dou spoke of this to Emperor Jing, who ordered an inquiry into its validity, and when it turned out to be so, bestowed rich gifts upon him. When Emperor Wen came to the throne, he ennobled Guangguo as Marquis of Zhangwu.... It was *his predetermined fate* that was responsible for his being both rich and of high rank.[141]

Wang thought the two key aspects of this kind of fate were that it was self-caused and numerical in character:

> Predetermined fate is the lord of luck both good and ill. It is the way of self-causation (*ziran zhi dao* 自然之道). It is the numbers for coincidence and conjuncture (*shi-ou zhi shu* 適偶之數). It is not caused to be as it is by response to the overwhelming pressure of extraneous things or other matter-vitality (*qi* 氣).[142]

'Number' (*shu* 數) was a property of such entities as dynasties:

> When the numbers of the Shang and the Zhou coincided with an ascendant [phase], the charismatic virtue (*de* 德) of their founders Tang and Wu coincided with [a phase of] abundance.[143]

This conception is close to the English folk-usage expressed in such phrases as 'his number was up' of someone seen as having been scheduled by fate for death. It is not culturally inaccessible to us, but equally should not be confused with more modern concepts of 'number'. In general, for Wang, "the numbers characterizing each period of time arrive of their own accord, and the behavior of human beings corresponds in conjunctural fashion" (*qishu zi zhi, renxing ouhe ye* 期數自至，人行偶合也).[144]

The most significant distinction was between a cause generated *within an entity*, which could on occasion be directed outward, or else respond to a signal from another entity, and *a coincidence between the trajectories* of two or more entities with a high level of contingency, even if not wholly accidental, if it may be put that way:

> If someone is crushed by a disintegrating house, or tumbles from a collapsing bank, this is not the vital essence (*jing* 精) of the house or the matter-vitality (*qi* 氣) of the bank killing him. It is that the house was old and the bank in ruins, the predestined fate of the person an evil one, and that their locations and trajectories *coincided* (*zaoju shilu* 遭居適履).[145]

He reiterates his conclusion: "if objects or phenomena (*wu-shi* 物事) encounter one another, and good or ill luck shares the same moment in time, this is conjunctural coincidence, not a response brought about by matter-vitality."[146] Luck was conceived of as *an inherent but varying property* of otherwise similar particular entities, not unlike color or weight:

> 'Predestined fate' indicates something that one is endowed with from the beginning. If people receive their disposition (*xing* 性) when they are born, they have received their predestined fate.... These are obtained together.[147]

Predestined fate commonly showed itself in people's faces and bodies, at least to those with the physiognomic skills to perceive it.[148]

He sharply restricts the agency of intentionality and consciousness in the universe. Heaven is unaware of what human beings are doing, and human matter-vitality is too slight to influence Heaven.[149] More trenchantly, "If Heaven has not been able to produce human beings deliberately (*gu sheng ren* 故生人), then, likewise, Its creation of the ten thousand things can not have been deliberate. The matter-vitality of Heaven and Earth joined together, and the ten thousand things conjuncturally arose of their own accord (*ouzi sheng* 偶自生)."[150]

Something that was about to happen also had an effect that spread out in front of it in the way suggested by the English commonplace 'coming events cast their shadows before'. Thus

> When a calamity is about to arrive, the body of its own accord has a strange presentiment. It is not something that can be activated by encountering other people. How can one prove this? — There might perhaps be an occasion when one happened upon a madman on the road who attacked one with his sword, but not necessarily with the thought of hurting one's body. In such a case one's own body would have a strange presentiment before the moment came.
>
> If one discusses the issue with this as the point of departure, the occurrence of the strange presentiment is the [foreshadowing] simulacrum of the inherent evil of the untoward event (*huobian zixiong zhi xiang* 禍變自凶之象), and is not caused by [the lunatic's] desire to hurt one. The person about to meet with misfortune will divine evil patterns of cracks by the turtle-shell oracle, and determine a baleful hexagram with the casting-stalks of the *Book of Changes*. When he steps forth from his doorway he will not see good omens, but portents of danger, and perceive the aethers (*qi* 氣) of disaster. The aethers of disaster will show themselves on his face in the way that white vapors round the solar disk and the evening star will appear in the Heavens.... Above and below there will be *coincidences* (*shiran* 適然) and self-caused *correspondences* (*zi xiangying* 自相應).[151]

Conversely, if, in the past, rain had fallen after Emperor Tang had prayed for it, this was not necessarily *because* he had prayed for it.[152] Correlation and causation had to be scrupulously distinguished.

Where would throwing-spills and dice have fitted into the picture sketched above? Although parts of the *Discourses Weighed in the Balance* are obsessed with chance they do not mention overtly aleatoric devices. It seems that there was no middle ground between wholly determined fate and wholly random contingency. Wang Chong was unaware that from seemingly chance individual events there could emerge regular collective patterns. Events had to be in one category or the other:[153]

> Changes in the Heavens in which the sun or the moon is diminished or occulted are a solar eclipse every 42 months and a lunar eclipse every 5 or 6 months.[154] There are constant numbers (*changshu* 常數) that determine eclipses. They are not [caused by the moral character of] politics.

In the section devoted to divination[155] he attacks the common notion that the procedures are ways of 'asking questions' of Heaven and Earth. Neither the tortoise undershell nor the milfoil stalks are "truly numinous and magical." Their reputation was due to the belief that they were remarkably old, and this belief derived from their names. Heaven and Earth were physically incapable of hearing questions from human beings, or of responding to them. He also dismisses the argument that the divining process is a way of accessing one's own thoughts:

> If someone uses his *numinous capacity* (*yong shen* 用神) to deliberate on a decision, but this deliberation is indecisive, causing him to turn to question the divining-stalks or the turtle's shell, and these then yield hexagram-numbers or a pattern on the shell that correspond with his own ideas, then this numinous capacity might be said to have given him clear instructions. At times, though, it may happen that his own thinking approves of a course of action that the shell-pattern omens or hexagram-numbers determine to be unlucky. Or again, the shell-pattern omens and hexagram-numbers are auspicious, but his own view is that the course of action is calamitous. Now, his thinking-process *is* his own numinous capacity; but to consider the shell-omens and the numbers as also [proceeding from] his own numinous capacity — the thoughts being within his breast and the omens and numbers outside it — is like there being no difference in the significance and circumstances of someone going indoors and sitting down or of his going outdoors for a walk! If the Spirits Bright create the shell-pattern omens and the hexagram-numbers, must these not be *different* from one's thoughts?

What is more, the shell and the dry stalks are dead. They cannot in and of themselves elicit a response from Heaven and Earth, which are alive. To the suggestion that the shell and stalks are like writing-tablets on which the commands of a ruler are inscribed, he replies that Heaven and Earth have neither mouths nor ears; being what they are of themselves without conscious effort (*ziran wuwei* 自然無為), they would not make an effort to answer human beings. Rather, the divination procedure used by the *Changes* creates a sort of microcosm:

> The 2-part separation of the stalks[156] is to make a likeness (*xiang* 象) of Heaven and Earth. Casting off stalks by 4 at a time (*si she* 四揲) is to create a likeness of the 4 seasons. Transferring the odd stalk to between the fingers separately[157] (*le* 扐) is to create a likeness of the intercalary month. This is simply determining the hexagram and its numbers (*gua shu* 卦數) by means of a method of likenesses that create congruences of categories (*yi xiang leixiang fa* 以象類相法). How can one describe it as Heaven and Earth responding to humans?

What happens if people use a frivolous, or, as we might say, 'experimental', approach?

> If someone were to knock on another person's door without having any question to put to him, or were to engage in empty disputation face-to-face without there being any problem to be resolved, the person principally concerned might laugh and not reply, or perhaps become angry and not answer. Suppose someone engages in turtle-shell divination or hexagram-determination, but pierces the turtle with no objective in mind, or counts off milfoil stalks without any substantial reason. *He will be fooling around with Heaven and Earth*, but will still get an omen or numbers. Will Heaven and Earth respond falsely to him?[158]
>
> Again, suppose that someone curses Heaven as he performs a turtle-shell divination, or pounds on the Earth as he handles the milfoil divination, acting in an utterly improper way. He will also obtain an omen and numbers. If one were to describe this as obtaining omens and numbers from Heaven and Earth, why is it that they don't extinguish his fire and scorch his hands, or make his fingers tremble and throw his numbers into confusion, or cause his body to suffer from sickness, and his blood-aethers to spurt in contention with each other? If, even so, they make

> an omen appear for him and numbers emerge, why is that that Heaven and Earth are not vexed by this, and bear no ill-will for their affliction?
>
> If one takes this as the basis for discussion, it is evident that *neither omen-taking with the turtle-shell nor milfoil-stalk divination is asking questions of Heaven and Earth*, nor are the shell-pattern omens or the hexagram-numbers responses from Heaven and Earth.

This also suggests, without actually saying so, that there had been occasions when people had experimented with the techniques, and had not suffered any retribution for doing so.

Appropriate outcomes only *seemed* to be caused by the good or bad luck that was inherent in each individual. In reality

> If one drills the turtle-shell, or counts off the milfoil stalks by fours, the omen-pattern of cracks and the hexagram-numbers come into existence of their own accord. Given the manifestation of omen-cracks and hexagram-numbers, [the prefigurations of] good and bad luck [likewise] come into existence of their own accord. But *human beings endowed with good and bad luck* encounter these in coincidental fashion (*shi yu xiangfeng* 適與相逢). Those endowed with good luck conjoin with omen-cracks telling of good fortune; those endowed with bad luck meet with hexagram-numbers foreshadowing ill-fortune. This is like a lucky person walking along a road and meeting with lucky events, and descrying objects that augur well; it is not brought about by the morally approving response (*ruiying* 瑞應) to him of lucky events and objects of good augury. The case is comparable for someone endowed with ill luck meeting with calamity on the road. *Good and evil do not make their appearance as the responses of Heaven*; they are *coincidental encounters* with good and evil [fortune].

'Coincidence' here is of course more than *mere* coincidence. The Chinese term *shi* 適 is also used for a bride going to the home of her fiancé to be married, so it carries the suggestion of 'a good match'. Other meanings include 'accident', 'opportune', and 'appropriate' — describing something that in the colloquial sense 'goes with' something else.

We can deduce from the foregoing that Wang Chong would most likely have regarded the outcome of each throw of a die as being, in his sense, self-caused, but the results as *also* involving the appropriate coincidence of the good or bad luck with which each player who tossed it was endowed. This second feature blocked the immediate path to probabilistic analysis by ruling out equiprobability of outcomes as between different players ostensibly doing the same thing, because *they* were not the same.

Wang is only one of several thinkers whose ideas need to be examined in this way. He is looked at here because he was unique among major Chinese philosophical figures in his combined preoccupation with chance events, and his attempt at intellectual rigor.

Conclusions

So far we have not found any unambiguous premodern written evidence of a Chinese concept of probability. This is in a social and economic context that would have seemed to call out for it: gambling, gambling-based purchasing, and numerically-based divination, much of it professionalized, in what was by later mid-imperial times a highly commercialized and monetized world. And one with competence in the basic mathematics needed. Both combinations and permutations were familiar, but not well enough comprehended to be linked to each other. There are, however, many indications that more may have been understood about odds and games-playing strategies under the cover of secrecy than was put into the public domain.

The cosmos was seen by not a few as an immense system of numbers, and there was a general preoccupation with chance and hazard; but even in the thought of Wang Chong, who seems to have been the most lucid of the philosophically obsessed in this domain, the notion of an inherent character of luckiness or unluckiness in individuals blocked off easy access to the idea of objective equiprobability as between different players. The basic technology of gaming in China had much in common with that of Europe and India, from which it borrowed and in turn contributed to Europe's repertoire, so in this respect

China was part of an intercommunicating Old-World games-playing region. The apparent lack of an exploratory serious playfulness, or experimentation, in such domains as dicing or *Book of Changes* divination, where it was easy to do simple ‘experiments’, is notable, yet there are stray hints that some may in fact have been tried.

Given the strongly favorable ‘external’ factors making for the development of a simple calculus of probability in China and its seeming absence before modern times, the conclusion has to be that, beyond some comprehensive tabulations of outcomes which were for ease of reference during play rather than for analysis, and some correct settings of fair payoffs in *implicit* terms of expectations (as we would see them), the ‘internal’ obstacles to abstract generalizing and analysis must have been a greater hindrance than appears likely at first sight. It is permissible to talk in such terms because in the domain of probability in advanced premodern commercialized societies, including the financial gains or losses in real-life gambling, there are only ‘right’ and ‘wrong’ answers, irrespective of cultural variations. The nature of these internal obstacles remains opaque for the moment, and in all likelihood they affected more in China than just the non-development of probabilistic thinking.

Against this has to be set the probability that this introductory survey has overlooked historical materials that can transform the picture suggested here. No one would be happier than the present author to see them identified, and to have been in some small measure the stimulus of their discovery.

APPENDIX A

The non-effect of forbidden bets on the fairness of *fantan*

To conceptualize what the forbidden bets mean in terms of expectations, consider 12 players playing simultaneously and in one scenario covering *all* the possibilities without any forbidden combinations, and in a second observing the rules given in the text. Assume that in all cases 1 is the number that comes up; the cases for the other numbers will follow the same pattern. Assume also that a player moving from the first to the second scenario will give ‘priority’ to keeping his principal number, where this applies, and will change his support number(s) in order to comply with the restrictions in the rules, or that he will alter his second choice in the case of option 3. Note that when a path bifurcates, the payoff for a single path has to be halved. On these assumptions we have something like the following:

Option 2

Player	*Bet*	*Payoff 1*	*Substitute Bets*	*Payoff 2*
1	**1**2	3		3
2	1**2**	1		1
3	**2**3	0		0
4	2**3**	0		0
5	**3**4	0		0
6	3**4**	0		0
7	**4**1	1		1
8	4**1**	3		3
Forbidden				
9	**1**3	3	**1**2	1·5
			14	1·5
10	1**3**	1	2**3**	0
			34	0
11	**2**4	0	1**2**	0·5
			23	0

Option 3

Player	*Bet*	*Payoff 1*	*Substitute Bets*	*Payoff 2*
1	12	2		2
2	21	2		2
3	23	0		0
4	32	0		0
5	34	0		0
6	43	0		0
7	41	2		2
8	14	2		2
Forbidden				
9	13	2	12	1
			14	1
10	31	2	32	0
			34	0
11	24	0	21	1
			23	0

<u>12</u>	2**4**	0	3**4**	0	*<u>12</u>*	42	0	41	1
			41	0·5				43	0
Totals		12		12			12		12

<u>Option 4</u>

Player	*Bet*	*Payoff 1*		*Payoff 2*
1	1**2**3	1		1
2	2**3**4	0		0
3	3**4**1	1		1
4	4**1**2	2		2
<u>Forbidden</u>			*<u>Substitute bets</u>*	
<u>5</u>	**1**23	2	4**1**2	2
<u>6</u>	12**3**	1	2**3**4	0
<u>7</u>	**2**34	0	1**2**3	1
<u>8</u>	23**4**	0	3**4**1	1
<u>9</u>	**3**41	1	2**3**4	0
<u>10</u>	34**1**	2	4**1**2	2
<u>11</u>	**4**12	1	3**4**1	1
<u>12</u>	41**2**	1	1**2**3	1
Totals		12		12

The substitute bets preserve the better's chances unchanged.

APPENDIX B

The yarrow-stalk divination procedure

Step **1**

The diviner takes 49 stalks and divides them as we would say 'at random' into 2 non-zero sets. This 'randomness' is where the cosmic resonances were thought to intervene, and hence it was not traditionally thought of as 'random', an idea only very roughly approximated by the nearest old Chinese term that might have been used, namely *ou* 偶, or *ouran* 偶然. The stalks are next placed in stores B and C. One stalk is then removed from B and put into store A. We can now say that A=1, B=48 - C, and C + B = 48 = 4 × 12. *Zero sets are excluded* since otherwise the matrix in Table 3 will not work.

We now determine two unique positive integers k and l such that one and only one of the following 4 equations holds for the given division of the stalks into 2 sets:

$$(B - 4k) + (C - 4l) = 1, \textit{ or } 2, \textit{ or } 3, \textit{ or } 4 \qquad \text{(B1).}$$

For example, if C = 31, then B = 48 - 31 = 17, and k = 4 and l = 7. In practical terms this was done by casting out sets of 4 stalks. We now prescribe:

$$B = B - 4k \qquad \text{and } C = C - 4l \qquad \text{(B2),}$$

that is, B (mod 4) and C (mod 4). In this illustrative case, we have A = 1 (as before), but B = 1, and C = 3. The echoes of modular arithmetic here are almost certainly genuine.[159]

The entry for step **1** in Table 3 is defined as A + B + C, which in this case is 5. This reduces the possible triples of numbers by half. *Only* the ($4k + 4l$) stalks removed by (B2) are used in step **2**. We may say that they are put into store R_1 , and that $R_1 := (4 \times 4) + (4 \times 7) = 44$.

If, however, C = 40, and B= 48 - 40 = 8. we have, by the same procedures, A + B + C = 9. This implies that in this case $R_1 = 4 + 36 = 40$. The calculation thus divides into two paths here according to the size of R_1.

Step **2**

The 44 (or 40) stalks in R_1 undergo the same procedures as the 49 stalks in step **1**. For example, for $R_1 =$ 44, if $C_2 = 29$, and $B_2 = 43 - 29 = 14$, we have $k = 3$ and $l = 7$, as well as B_2(mod4) = 2 and C_2(mod 4) = 1, so

(14 - (4 × 3)) + (29 - (4 x7)) = 2 + 1 = 3 , and $A_2 + B_2 + C_2 = 4$, and $R_2 = (4 \times 3) + (4 \times 7) = 40$.

The ‘4’ goes into the second column of Table 2. There are now only 2 triples possible, <1>, the Old Yang, or a <*i*>, the second of the Young Yins. Which of these two will be determined in step **3**.

For $R_1 = 40$, we might have $C_2 = 32$, and $B_2 = 39 - 32 = 7$. Then (7 - (4 × 1)) + (32 - (4 × 7)) = 3 + 4 = 7, and $R_2 = 4 + 28 = 32$. Since $A_2 + B_2 + C_2 = 8$, we have the triple (9,8,?), which can only become either <-1> or the first of the <- *i* >s once the third step has been performed.

Step **3**

The same procedures are applied again to the 40 (or 36 or 32) stalks in R_2. Continuing with our first example, we select $C_3 = 26$, so $B_3 = 39 - 26 = 13$. Since (13 - (4 × 3)) + (26 - (4 × 6)) = 1 + 2 = 3, we have the sum $A_3 + B_3 + C_3 = 4$. The line selected is therefore that which is assigned in Table 3 to the triple (5,4,4), namely <+1>, the Old Yang.

Continuing our second example, where $R_2 = 32$, we might have $C_3 = 27$, with $B_3 = 31 - 27 = 4$. Then B_3= 4 - 0, and C_3 = (27 - (4 × 6)) = 3. $A_3 + B_3 + C_3 = 1 + 4 + 3 = 8$, and we have (9,8,8) which is assigned to <-1>, the Old Yin.

These operations are repeated 5 more times to generate the rest of the hexagram.

It is handy, for following all this, to have a table showing the numbers of stalks used in each row at the start of each of the steps. This is provided in Table B1.

TABLE B1

The numbers of stalks at the start of each step, with positions corresponding to those in Table 3.

Steps	**1**	**2**	**3**
	49	44	40
	49	40	32
	49	40	32
	49	44	36
	49	40	36
	49	40	36
	49	44	40
	49	44	36

Armed with this, generate Table 4 in the main text by replacing the entries in the 3 ‘step’ columns of Table 3 with probabilities and then multiply the 3 probability entries along each row by each other to determine the overall probability of each of the paths to a particular type of line. This requires an examination of the frequencies of the occurrences of the values of $A_{1,\ 2,\ \text{and}\ 3}$, $B_{1,\ 2,\ \text{and}\ 3}$, and $C_{1,\ 2,\ \text{and}\ 3}$ once the multiples of 4 have been subtracted, leaving one of the four residues 1, 2, 3, and 4.

Taking step **1** as an example, it is clear that, since zero sets are forbidden, the highest possible value for B is 47 (with C = 1) and the lowest is 1 (with C = 47). When the multiples of 4 have been deducted, there is *one less* occurrence of 4 (which appears 11 times) than of 1, 2, and 3 (which appear 12 times each). The same is true for the values of C once the multiples of 4 have been subtracted. The pairs of residual values of B and C are mutually interdependent: a 4 in one implies a 4 in the other, and so on. The B + C pairs with a value of (4 + 4) have a frequency of 11/47, whereas (1 + 3), (2 + 2), and (3 + 1) each occurs with frequency of 12/47. [160] Analogous considerations apply in steps **2** and **3**, but require a little more care on account of the differing R values.

APPENDIX C

Assigning shares of stake-money to an unfinished game of *Top The List*

The first 2 throws, A_1 and B_1, are straightforward. If A thows a 4, 5, or 6 at A_1, he wins at once with a score of 16, 17, or 18. The probability of this happening is 3 × 1/6, or one half. If, however, A has thrown a 1, 2, or 3, his score will be 13, 14, or 15. At B_1, B — with 10 points — therefore has a 1/6 chance of winning in any of these 3 cases by throwing a 6. In the context of the *whole* game, which must include an appropriate preceding score at A_1, these have a combined probability of 3 × 1/6 × 1/6 = 3/36. The probabilities of winning for each of the 2 players thus far into the analysis are therefore respectively 18/36 and 3/36.

There is a sharp rise in difficulty at the next step. This is often a cause of psychological blockage in attempting to solve a problem.[161] We therefore become slightly more formal and adopt the notation 'A's score : B's score / ... ;' and represent a given path of scores through time as, for example, '12:10; 13:10 / 13:16', the semi-colon denoting the end of a throwing cycle, and the solidus the division between A's and B's turns. We also use '*x* — *y*' to represent 'the (integer) values from *x* through *y*'.

Immediately before A_2, A's score in the 3 × 5 = 15 paths still continuing, since the won games have stopped, is 13, 14, or 15. Each of these is matched with values 11 — 15 for B, since B won if he threw a 6 and so reached 16 on B_1. Since A_2 can yield 1 — 6, the pairs of scores now become 3 blocks each of 5 × 6 = 30 outcomes, namely (i) 14 — 19 : 11 — 15, (ii) 15 — 20 : 11 — 15, and (iii) 16 — 21 : 11 — 15. In (i) A wins outcomes 16 — 19 : 11 — 15, in (ii) he wins 16 — 20 : 11 — 15, and in (iii) 16 — 21 : 11 —15. These are 20 + 25 + 30 = 75 out of 90 paths. In the context of the whole game, these represent a probability of $75 \times (1/6)^3$ = 75/216.

The 15 paths still continuing for B_2 are 1 block of 14: 11 — 15, and 2 blocks of 15 : 11 — 15. In each of these 3 blocks, from a starting score of 11 — 15, throws respectively of 5 — 6, 4 — 6, 3 — 6, 2 — 6, and 1 — 6 give B victory. Together these successful outcomes sum to 3 × 20 = 60, and their probability in the context of whole game is $60/(1/6)^4$ = 60/1296.

There are 3 × 10 = 30 paths continuing for A_3. In 1 block of 10 A has 14 points, and in the other 2 a score of 15. These each yield 10 × 6 = 60 outcomes. A wins all the 2 × 60 =120 stemming from his scores of 15. He wins 10 × 5 = 50 of these cases where his score is 14, only failing to win in the 10 cases when he throws a 1. This turn adds $170/(6^5)$ = 170/7776 to his probability of winning.

Ten paths remain for B_3: 15:12, 2 × (15:13), 3 × (15:14), and 4 × (15:15). He wins 4 × 6 of his 15s, 3 × 5 of his 14s, 2 × 4 of his 13s, and 1 × 3 of his 12s. These add $50/(1/6^6)$ = 50/46,656 to his chances of winning.

A wins the last 10 paths still continuing at A_4, adding $10/(1/6^7)$ = 10/279,936 to the probability of his success. A thus has an 87% probability of winning, and B 13%.

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[1] Bashmakova and Smirnova, *Algebra*, chapter 1.

[2] Li Yan (or ‘Nian ‘), *History of Calculation in China.*

[3] Needham, with Wang Ling, “Mathematics”.

[4] Libbrecht, *Chinese Mathematics in the Thirteenth Century:*.

[5] Martzloff, *Chinese Mathematics.*

[6] Volkov, *Sous les nombres, le monde*.

[7] Nakayama Shigeru, *Academic and Scientific Traditions in China, Japan, and the West*.

[8] Crombie, *Styles of Scientific Thinking*.

[9] Elvin, “The Man Who Saw Dragons,” *passim*.

[10] See Hacking, *The Emergence of Probability*, id., *The Taming of Chance*, and Gigerenzer et al., *The Empire of Chance*. Von Plato, *Modern Probability* , has an astonishing supplement on Nicole Oresme’s anticipation of aspects of geometric probability in the fourteenth century.

[11] Martzloff, *Chinese Mathematics*, 55.

[12] Guo and Xiao, *Chinese Gambling*, 122-3. I am very much indebted to Professor Li Bozhong for the gift of a copy of this book.

[13] Guo and Xiao, *Chinese Gambling*, 81.

[14] Guo and Xiao, *Chinese Gambling*, 42 (fourth century CE under the Easten Jin).

[15] The translation ‘slaves’ is warranted by Hong’s own gloss that “they call ‘male and female slaves’ those men and women whom they have seized and sometimes sell off to others.” Hong Mai, “Pu shuang xu” in *Shuofu*, *j*. 102.

[16]Guo and Xiao, *Chinese Gambling.*

[17] Song Lian et al., *History of the Yuan*, V, *j.* 105, xingfa iv, 1685.

[18] Parlett, *Board Games*, 27-8, mentions cubical dice as being known to the ancient Lydians, Greeks, and Etruscans.

[19] On this last see Guo and Xiao, *Chinese Gambling*, 63 and plate 2. It was essentially a sphere with 18 flattened faces. There were 3 ‘great circle’ circuits. That around the ‘equator’ was **3**-12-*1*-11-**2**-7-*5*-10-**3** etc, summing to 51, where the bold-type digits indicate the faces shared with the one of the other circuits and the italicized digits those shared with the other. The other circuits had two characters rather than a numeral at the ‘north pole’ and ‘south pole’. The first of these was *jiao* 馬喬 ‘a high-spirited horse’ and the second *wei* 妻畏, probably meaning ‘a gentle wind’ (assuming it was a variant of the 委畏 given in the Kangxi dictionary). The cycles were *jiao*-13-**2**-9-*wei*-6-**3**-15-*jiao* etc, and *jiao*-14-*5*-4-*wei*-8-*1*-16-*jiao* etc, both summing to 48. The version in Li Ling, *Magical Techniques*, 168, is different: all 3 circuits run through *jiao* and *wei*, and no locations ar shown for 13, 14, 15, and 16.

[20] D. Parlett, *Board Games*, 22-3.

[21] Namely 5B or 5W, (4B + 1W or 4W + 1B, and 3B + 2W or 3W + 2B. The probability for a 5 of one or other of the colors was 2 × 1/32, that for one or other of the 4-+-1 groups was 2 × 5/32, and for a 3-+-2 it was 2 × 10/32.

[22] Cheng Dachang, “Chupu jinglüe” in *Shuofu*, *j.* 102.

[23] Deng Ai was a general who lived in the third century CE. I have found no reference to this book, if book it is. The sentence as it stands makes no sense. There was confusion in later times over whether *xiao*, the 'unfilial owl,' was a high or low score. See *Gezhi jing yuan* , 2671. Li Ling, *Magical Techniques*, 168, states that in Sixes, which was an older game that *chupu* and even less well understood today, the owl was a promoted piece, which implies that it would tend to be victorious. Alternatively, *sheng* 勝 can be corced into the passive sense of 'is conquered', which makes for good sense but poor grammar.

[24] Cheng Dachang, "Chupu jinlüe" 7a, in *Shuofu*.

[25] The *Taiping yulan*.

[26] The word is *qiong* or *jing*, written 煢 usually with 几 as the lower element, or else as 橩. See the discussion in Guo and Xiao, *Chinese Gambling*, 19-20.

[27] Yan Zhitui, *Family Instructions*, 372.

[28] *Gezhi jing yuan*, 2675.

[29] *Gezhi jing yuan*, 2673. The source is the *Hou Wei Li Shao xu* 後魏李邵序, the 'Preface by Li Shao of the Later Wei'. The 'Later Wei' is the Toba Wei dynasty, of the fifth and sixth centuries CE. Cp also *Gezhi jing yuan*, 2679 for much the same point.

[30] Cited in Guo and Xiao, *Chinese Gambling*, 60. The translation is at some points problematic.

[31] Western Zhu (*Xi Zhu* 西竺), India being *Tianzhu* 天竺.

[32] In the third century CE.

[33] *Shuofu*, *j*.102, "Pi shuang xu", 1a.

[34] Described in Parlett, *Board Games*, 42-8.

[35] Guo and Xiao, *Chinese Gambling*, 135-8.

[36] Feyerabend, *Farewell to Reason*, 88.

[37] Elvin, "Blood and Statistics."

[38] Elvin, "The Man Who Saw Dragons," 26-8.

[39] Li Kuangyi, *Zixia lu*, in *Qinding siku quanshu* vol. 850, *zhong*, 156. The Chinese text is provided here as a safeguard:卜則嫗非卜筮者必話桑道茂之行有嫗一無所知大開小肆自桑而卜回者必日嫗於桑門賣卜其神乎俾來覆之桑言休則嫗言咎桑言咎則嫗言休顧後中否桑嫗各半或有折話者日斯管公明門前嫗也咸誤矣案符子云齊有好卜者十而中五鄰人不好卜常反之亦十中五與不卜等耳蓋是子家設理之詞後人呼聲而至是愚欲歸實故證之.

[40] A play on words, since *sangmen* 桑門 'monk' sounds as if it also meant 'mulberry gate'.

[41] *Shuofu*, 19:ab, "Zeng Sanyi yinhualu." Zeng lived in the late twelfth and early thirteenth centuries.

[42] Professor Nakayam Shigeru kindly wrote to me on October 16, 2000, that, according to his researches, "As far as I know in the field of astronomy, Tang China started a complicated method of interpolation and so it is simply unthinkable for them not to know simple arithmetical means." The point that this ability is different from the capacity to conceptualize a *distribution* is made by D. G. Rees's comment on the old joke that a statistician is someone who, on plunging one foot into a bucket of boiling water and the other into a bucket of melting ice, declares 'On average, I feel just right!' The purpose of statistics, Rees observes, "is to...analyse data which *vary*." Rees, *Essential Statistics*, 37.

[43] I am grateful to Professor Jochi Shigeru for drawing this passage to my attention.

[44] Bai Shangshu, *Nine Chapters*, 21-3; Qian Baozhong, *Ten Classics*, 'Nine Chapters', 71-2; *History of Chinese Mathematics*, II.59-60. On text of the Nine Chapters see Martzloff, *Chinese Mathematics*, 127-36.

[45] Qian Baozong, *Ten Classics*, 'Nine Chapters', 6, 1-2. Bai Shangshu, *Nine Chapters*, 184-7 gives a thorough analysis of the method. See also *History of Chinese Mathematics*, II.85-6.

[46] The exact meaning of the terms in single quotes in this sentence is not relevant here, and they should be taken as approximations only.

[47] Yan Zhitui, *Family Instructions*, 365-6. Emphasis added. There is an English translation by S-Y. Teng, *Family Instructions for the Yen Clan* The translation he gives of this passage on his p. 203 differs in many respects from the one I have given here. The Chinese is not easy and I would not claim necessarily to be more correct, only a little more coherent.

[48] On hexagrams see the later section on divination.

[49] Computer simulation programed on MacPerl by the author.

[50] Epstein, *Theory of Gambling*, 3-4.

[51] Shen Gua, *Notes from the Garden of the Brooklet of Dreams*, 181-2. See also Martzloff, *Chinese Mathematics*, 98.

[52] For those unfamiliar with *go*, it may be useful to add that diagonals play no part in the game. A piece away from the edges can be 'surrounded' by 4 enemy pieces, one on each of the 4 lines leading away from the intersection where it is placed.

[53] *Mencius*, I.i.vii. 21. As Dr Garry Tee of Auckland University points out, the sage had never seen a mudskipper climbing a mangrove tree in a Queensland creek.

[54] Li Kuangyi, *Zixia lu*, *zhong*, in the *Qingding siku quanshu*, v. 850, 157.

[55] Luo Xinben and Xu Rongsheng, *Gambling Customs*, 100-01.

[56] Guo and Xiao, *Chinese Gambling*, 181-3.

[57] This reconstruction is based on the most coherent compromise I can devise from the somewhat conflicting accounts in Guo and Xiao's *Chinese Gambling*, and in Luo and Xu's *Gambling Customs*.

[58] We have also to consider the possibility that the rules may have specified that the principal had to be the first of the pair. In other words: 2´. Permitted: **1**2, **2**3, **3**4, and **4**1. *Not permitted:* 1**2**, 2**3**, 3**4**, or 4**1**; **1**3, 1**3**, **2**4, or 2**4**.

[59] Pascal's triangle starts with a '1' at its summit, and each of the numbers in each row below is derived as the sum of the two numbers offset half a position to each side of it in the row above: (1), (1, 1), (1, 2, 1), (1, 3, 3, 1), etc. The numbers in each row sum to 2 raised to the power of the rank of the row, starting with 0: thus $\sum(1, 3, 3, 1) = 2^3 = 8$.

[60] Martzloff, *Chinese Mathematics*, 17, 230-1, 304-5.

[61] Luo and Xu, *Gambling Customs*, 106-08.

[62] Li Kuangyi, *Zixia ji*, in Wang Yunwu, ed., 16. Also in *Qinding siku quan shu*, v. 850, *zhong*, 157.

[63] On which see the introductory section.

[64] Li Yanshou, *History of the Southern Dynasties*, *j.* 53, 1312.

[65] Literally 'boys' (*xiao er* 小兒), but it is obvious from the context that these were not children.

[66] Chou Zhaoao, *Du Fu*, *j.* 15, "Kuizhou ge shi jueju," 120. I have supplied the word 'merchants', following one of the commentaries.

[67] Meng Yuanlao, *Dream of the Glories of the Eastern Capital*, 162. I have also consulted the Japanese translation by Iriya Yoshitake and Umehara Kaori, *Tôkyô muka roku*, 199.

[68] Note cited by Deng Zhicheng in Meng Yuanlao, *Dream of the Glories of the Eastern Capital*, 164.

[69] Note cited by Deng Zhicheng in Meng Yuanlao, *Dream of the Glories of the Eastern Capital*, 163.

[70] Note cited by Deng Zhicheng in Meng Yuanlao, *Dream of the Glories of the Eastern Capital*, 163-4.

[71] Luo Xinben and Xu Rongsheng, *Gambling Customs*, 106.

[72] Li Dou, *Yangzhou huafang lu* [The painted barges of Yangzhou], 16, "Shu gang lu," cited by Deng Zhicheng in Meng Yuanlao, *Dream of the Glories of the Eastern Capital*, 164-5.

[73] Or, possibly, 'Gambling By Tens'.

[74] Cheng Dachang, "Chupu jinglüe," in *Shuofu*, 102:3b-4a.

[75] Probably what is normally known as the 'Northern Wei' or 'Western Wei'. The Northern Zhou did not have an Emperor Wen.

[76] Cheng Dachang, "Chupu jinglüe," 102:3b, in *Shuofu.*

[77] Who reigned from 494 to 498 CE.

[78] This translation is a guess for the term *houxiang* 候相.

[79]There is a brief history in Wu Kang, *Outline of the* Changes of the Zhou. For the general background, see Needham, with Wang Ling, "History of Scientific Thought."

[80] R. Wilhelm, *Changes,* I.xl-xli.

[81] See H. Wilhelm, *Change*, and Elvin, "Was there a transcendental breakthrough in China?," 295-99.

[82] See Li Ling, *Magical Techniques,* chapter 4.

[83] Wu Gang, *Outline of the* Changes of the Zhou, 37.

[84] Situations arising from the interaction of two persons or trends could be represented by the 'products' of 2 hexagrams derived by multiplying corresponding elements. Thus the 'product' of #12 *bi* 'standstill' with old yin at the bottom and old yang at the top <- 1, *i*, *i*, - *i*, - *i*, 1> with #49 *ge* 'revolution' with old yin in second place and old yang in fifth place =<- *i*, -1, - *i*, - *i*, 1, *i* > can be expressed as $<{}_2ge_5> \times <{}_1bi_6> = {}_4sun_3$, which is #57 *sun* 'the gentle' <*i*, - *i*, 1, - 1, - *i*, *i* >, with changing lines in the third and fourth places. The first of these indicates 'humiliation' but the last one stands for 'remorse vanishes'. (See R. Wilhelm, *Changes*, I, 236-7.) Hexagram multiplication and division of this sort form a closed commutative system. Multiplying a hexagram by its 'dynamic complement', in which *i* and 1 have been interchanged produces standstill, with every component equal to *i*. Multiplying a hexagram by its 'charge complement', in which yin and yang have been interchaged, produces maximal mutation as every component becomes a changing one, with value 1. The historical absence of any manipulative experimentation of this sort suggests a lack of abstract imaginative power. Isn't it an 'obvious' question to ask, How did the hexagrams divined for two interacting persons, each with a different problem, interact? Needham and Wang, *Science and Civilisation in China*, II (1956), section 13, 335. speak of the "the devastating effects of the Book of Changes" on the development of scientific thought. This is surely a defensible position, but my own feeling is that the crucial failure may have been not pursuing its possibilities and implications *far enough*. This would have led to the point at which, inevitably, the system would have broken down as self-evidently inadequate, and compelled a rethink.

[85] Cheng Shiquan, "The transformation of the *Changes of Zhou* into hexagrams," in Huang Shouqi and Zhang Shanwen, II, 377.

[86] Needham, with Wang Ling, "Mathematics", 81-90; Martzloff, *Chinese Mathematics*, 104, 192-7. The greater part of the 'Fang tian' chapter in *The Nine Chapters* (see Bai Shangshu, and Qian Baozong) deals with fractions.

[87] R. Wilhelm, I.xxxiv.

[88] For overviews, see Wu Kang, *Master Shao's Study of the* Changes, and Cai Dean, *Evaluation of the* Study of

the Changes of Heaven *by Shao Kangjie*. The most important text is Shao Kangjie [Yong], *Direction of the World by the Supreme Ultimate*.

[89] Ban Gu, *History of the Han*, 65, 'Biography of Dongfang Shuo', 2843-4.

[90] See *Cihai* , *shefu*, 433.

[91] *Lunheng*, 600.

[92] Sometimes given as *ptarmica* or *sibirica* by earlier writers.

[93] I follow the formulation given by R. Wilhelm, *Changes*.

[94] There is a version of this table, with other matter, in Wang Qi, *Illustrated Encyclopedia of the Three Principles*, vol. 4, *renshi* 10:18ab, 1795-6. The three differing derivations of the young yin and young yang are each seen as having a distinctive nature. They are each described by one of the hexagrams that is the reduplicated form of a trigram. Thus (9, 4, 8) is symbolized by *kan* 坎 'the abysmal'.

[95] Needham, with Wang Ling, "Mathematics', 140, hint at the existence of "an esoteric doctrine," without elaborating further.

[96] On 'random division' see Appendix B.

[97]Guo Shuanglin and Xiao Meihua, *Chinese Gambling*.

[98] Cited in the *Hu Yinglin bicong* [The jottings of Hu Yinglin] cited in *Gezhi jing yuan*, 2684.

[99] *Gezhi jing yuan*, 2655-6.

[100] *Gezhi jing yuan*, 2655.

[101] *Gezhi jing yuan*, 2651 and 2660.

[102] *Geizhi jing yuan*, 2655. 'Position-power' is *shi* 勢, and to be contrasted with mere brute strength.

[103] *Gezhi jing yuan*, 2654.

[104] *Gezhi jing yuan*, 2654.

[105] Chinese spirits were thought to travel in straight lines.

[106]*Gezhi jing yuan*, 2651.

[107] Who was thought, according to one legend, to have used the game to instruct his son.

[108] *Gezhi jing yuan*, 2653-4.

[109] On coloured cards meetings see Guo Shuanglin and Xiao Meihua, *Chinese Gambling*, 185-9, 331-40, and Chen Dingshan, *Ancient Hearsay from Shanghai*, 62-4.

[110] Yongnuo jushi, *Things close enough to hear,* 7:6ab, 4494.

[111] Translation tentative.

[112] If it was returned, the expected return would have been $0 \cdot 96875x$, and the loss each round just over 3 per cent, closer to the level in European casinos' version of roulette.

[113] Epstein, *Theory of Gambling*, 113.

[114] Chen Dingshan, *Ancient Hearsay from Shanghai*, 63-4.

[115] Epstein, *Theory of Gambling*, 115.

[116] A very tentative translation of what is more literally 'the rule was 90 per cent went into the roller'.

[117] The common phrase for being the winning name.

[118] *Cihai*, 877.

[119] James Greenbaum (A.N.U.), personal communication based on observation in Taiwanese bars.

[120] Xie Zhaozhe, *Fivefold Miscellany*, 500-01.

[121] Epstein, *Theory of Gambling*, 101-03.

[122] That is, one whose transpose (or row/column interchange) changes sign

[123] Epstein, *Theory of Gambling*, 102. There are 3 further sets that are 'basic' in the sense of not being linear combinations of each other.

[124] See, for example, Xie Zhongsan and Ouyang Yi, *Sino-Western Calendar*.

[125] The discussion that follows is based on the translations in David, *Games, Gods and Gambling*, 239-49.

[126] Guo Shuanglin and Xiao Meihua, *Chinese Gambling*, 179-80.

[127]*Shuofu*, *j*. 101, 'Da ma tu'.

[128] *Shuofu*, *j*. 102, 'Chuhong tu'.

[129] It appears in the 1652 edition of *Shuofu j*. 102, and this fixes the latest possible date. The meaning of the name is not clear.

[130] *Sunzi xu* 孫子序, cited in Pan Zimu, *Jizuan yuanhai*, *j*. 66, "Wuli ying-xu," 4315.

[131] Keightley, *The Ancestral Landscape*, *passim*..

[132] Elvin, "Was there a Transendental Breakthrough?"

[133] Elvin, "Who Was Responsible for the Weather?"

[134] We would calculate it as 6!/(3! x 3!).

[135] Examples in H. Wilhelm, *Change*, figures 1 and 2.

[136] References are to Huang Hui, ed., *Lunheng*.

[137] *Lunheng*, 624.

[138]*Lunheng*, 36.

[139] *Lunheng*, 37.

[140] *Lunheng*, 51.

[141] *Lunheng*, 86-7.

[142] *Lunheng*, 91.

[143] *Lunheng*, 91.

[144] *Lunheng*, 93.

[145] *Lunheng*, 94.

[146] *Lunheng*, 95,

[147] *Lunheng*, 116-7.

[148] *Lunheng*, 101-112.

[149] *Lunheng*, 195-6, 221.

[150] *Lunheng*, 137.

[151] *Lunheng*, 225.

[152] *Lunheng*, 225.

[153] *Lunheng*, 773.

[154] Determining how near these numbers came to a good estimate under the historical circumstances would need to take into account at least the following points: (1) The area concerned, since solar eclipses are visible only along a limited path on the Earth's surface; (2) the likely frequency of visibility, which would have depended on the weather; and (3) the degree of partiality. The Han theory of a period of 135 months with 23 lunar eclipses is equivalent to roughly one every 5· 9 months. (See Needham and Wang, *Science and Civilisation in China*, II, 421.) According to Dr Garry Tee of Auckand University, "from any fixed place on Earth, an annual average of about 2 lunar eclipses can be seen if the sky is clear. And the solar eclipses visible from all possible places within the Han Empire might average something like 1 every 42 months" (personal communication 27.x.00).

[155] *Lunheng*, 994-1004. All the quotations that follow are taken from these pages.

[156] Into the B and C stores described in Appendix B.

[157] Into store A described in Appendix B.

[158] And if not, why not? He deserves no better. But falsehood is not in their natures.

[159] For an introductory discussiom of the 'Chinese Remainder Problem', see Blunden and Elvin, *Cultural Atlas of China*, 197 in the section "Principles of mathematics'. This problem is referred to by the fourth century CE.

[160] The point made in this paragraph has been overlooked by others who have made somewhat similar calculations. See, for example, E. A. Hacker, *The* I Ching *Handbook*.

[161] A point made to me in Cambridge long ago by the late Dr Ian Hughes, who later went to the Department of Psychology at Aberdeen University. The stimulus was the well-known 'yak' problem, represented by 2 teams of 4 nose-to-tail kitchen matches going in opposite directions along a track one yak-width (or match-width) wide. On one side of the track there is a bottomless gorge, and on the other an unclimbable precipice. The leaders stop and face each other one yak's-length (or match-length) apart. Being 'progressive' animals they are only allowed to move forward, or to jump over one yak (facing either way) into a space beyond — the 'great leap forward'. Can the 2 teams work out a way to pass each other while observing these rules and without falling off the track? The answer is 'yes' but the level of difficulty rises sharply after the simple start.